New representations for (max,+)-automata with applications to the performance evaluation of discrete event systems

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Abstract: A large class of timed discrete event systems can be modeled thanks to (max,+)-automata, that is automata with weights in the so-called (max,+) algebra.
In this contribution, new representations are proposed for (max,+)-automata. Indeed, specific recursive equations over (max,+) and (min,+) algebras are shown to be suitable for describing extremal behaviors of (max,+)-automata. It is underlined that several performance evaluation elements, such as maximum and minimum string execution times, can be easily derived from these representations.

Keywords: discrete event system, performance evaluation, (max,+) automaton.

1. INTRODUCTION

Several formalisms have been introduced and experienced for studying Discrete Event Systems (DES). Some of them rest on models defined over a dioid (idempotent semiring) such as (max,+) algebra Baccelli et al. (1992). In particular, Stéphane Gaubert has first shown in Gaubert (1995) that a large class of DES can be studied by means of so-called (max,+) automata, that is weighted automata with weights (multiplicities) in (max,+) algebra. Their behavior is represented by formal power series with coefficients in (max,+) algebra which play an equivalent role to languages for logical (boolean) automata. Unfortunately, it has been shown that in general (for nondeterministic automata) there are important problems that are undecidable (or whose decision status is unknown). In particular, equality and inequality of two rational (max,+) formal power series is undecidable, see Krob (1994). This compromises the applicability of results to general systems. For example, to expect the realizability of controllers the supervisory control approach in Komenda et al. (2009b) has been restricted to deterministic automata, and synchronous product has been defined in Komenda et al. (2009a) in order to consider the control of complex (nondeterministic automata) in a decentralized manner.

Motivated by this observation, the present contribution proposes alternative representations for (max,+) automata. These representations describe the behavior of automata less accurately (only extremal behaviors are described) but it is hoped that thanks to these problems of significant interest can be tackled with reasonable complexity. More precisely, we define specific recursive equations over (max,+) and (min,+) algebras in order to describe so-called worst-case and optimal-case behaviors of the automata. These have direct applications to evaluate the performances of the systems. In particular, indicators such as maximum and minimum execution times can be derived or approximated and these results are discussed regarding the related works in the literature Gaubert (1995), Su and Woeginger (2011). We also propose some refinements of these results as well as new possible indicators.

This paper is organized as follows. In the next section, preliminaries on dioids are recalled together with (max,+) automata and their properties. In Section 3, the new representations for (max,+) automata are introduced. These naturally lead to some performance evaluation elements described and compared with related results in the literature. These considerations are the topics of Section 4. A conclusion and some prospects are given in Section 5.

2. PRELIMINARIES

2.1 Dioids

Necessary algebraic concepts on dioids are briefly recalled in this section (see the monographs Baccelli et al. (1992) and Heidergott et al. (2006) for an exhaustive presentation).
A *dioid* is a *semiring* in which the addition $\oplus$ is idempotent. The addition (resp. the multiplication $\otimes$) has a unit element ε (resp. e).
Example 1. The set $\mathbb{R} \cup \{-\infty\}$ (resp.$\mathbb{R} \cup \{+\infty\}$) with the maximum (resp. the minimum) playing the role of addition and conventional addition playing the role of multiplication is a dioid, denoted $\mathbb{R}_{\max}$ (resp.$\mathbb{R}_{\min}$) , with $e = 0$ and $\varepsilon = -\infty$ (resp. $\varepsilon = +\infty$)) and is usually called (max,+) algebra (resp. (min,+) algebra).

The set of $n \times n$ matrices with coefficients in dioid $\mathbb{R}_{\max}$ (resp. $\mathbb{R}_{\min}$), endowed with the matrix addition and multiplication conventionally defined from $\oplus$ and $\otimes$, is also a dioid, denoted $\mathbb{R}_{\max}^{n\times n}$ (resp. $\mathbb{R}_{\min}^{n\times n}$). The zero element for the addition is the matrix exclusively composed of ε. We denote I_n the neutral element of the multiplication, which is the matrix with e on the diagonal and ε elsewhere.

A matrix $A \in \mathcal{D}^{n\times n}$ ($\mathcal{D}$ denotes $\mathbb{R}_{\min}$ or $\mathbb{R}_{\max}$) is said to be irreducible if and only if $\exists m \in \mathbb{N}$ such that $[A^m]_{ij} \neq \varepsilon$ for all $i, j = 1 \ldots n$.
An irreducible matrix admits a unique eigenvalue $\lambda \in \mathcal{D}$ and possibly several eigenvectors $v \in \mathcal{D}^{n\times 1} \backslash \{\varepsilon\}$ such that

$$A \otimes v = \lambda \otimes v.$$

For an irreducible matrix $A \in \mathcal{D}^{n\times n}$ of eigenvalue λ, we have

$$\exists N, c \in \mathbb{N} \text{ such that } \forall n \geq N, A^{n+c} = \lambda^c \otimes A^n.$$

The smallest integer c is called *cyclicity* of A.

A reducible matrix A can admit several eigenvalues, and the approach consists in decomposing the matrix into several irreducible blocks to study its cyclicity (see Gaubert (1997)).

2.2 (Max,+) automata

Automata with multiplicities in the $\mathbb{R}_{\max}$ semiring are called (max,+) automata (see Gaubert (1995) or Gaubert and Mairesse (1999) for more complete introductions).
A (max,+) *automaton* G is a quadruple (Q, Σ, α, μ) where [1]

- Q and Σ are finite sets of states and of events;
- $\alpha \in \mathbb{R}_{\max}^{1\times|Q|}$ is such that $\alpha_q \neq \varepsilon$ if q is an initial state;
- $\mu : \Sigma^* \to \mathbb{R}_{\max}^{|Q|\times|Q|}$ is a morphism specified by the matrix family $\mu(a) \in \mathbb{R}_{\max}^{|Q|\times|Q|}$, $a \in \Sigma$, knowing that we have for a string $w = a_1 \ldots a_n$
$$\mu(w) = \mu(a_1 \ldots a_n) = \mu(a_1) \ldots \mu(a_n),$$
where the matrix multiplication involved here, is the one of $\mathbb{R}_{\max}^{|Q|\times|Q|}$. A coefficient $[\mu(a)]_{qq'} \neq \varepsilon$ means that the occurrence of event a causes a state transition from q to q'.

A (max,+) automaton is said to be *deterministic* if

- it has a unique initial state, namely, there is a unique $q \in Q$ such that $\alpha_q \neq \varepsilon$;
- from each state, the occurrence of an event can not induce the occurrence of several possible state transitions, namely, if for all $a \in \Sigma$ each line of $\mu(a)$ contains at most one element not equal to ε.

Example 2. Figure 1 is an example of graphic representation which can be associated with every (max,+) automaton:

- the nodes correspond to states $q \in Q$;
- an edge exists from state $q \in Q$ to state q' if there exists an event $a \in \Sigma$ such that $[\mu(a)]_{qq'} \neq \varepsilon$: it represents the state transition when event a occurs and the value $[\mu(a)]_{qq'}$ is interpreted as the duration associated to a (namely, the time activation of event a before it can occur) ;
- an input edge symbolizes an initial state.

For this example, we have $Q = \{I, II\}$, $\Sigma = \{a, b\}$, and

$$\alpha = (\, e \;\; e \,)\,,\;\; \mu(a) = \begin{pmatrix} \varepsilon & 3 \\ \varepsilon & 2 \end{pmatrix}, \mu(b) = \begin{pmatrix} \varepsilon & \varepsilon \\ 2 & \varepsilon \end{pmatrix}.$$

[1] to simplify the presentation and without loss of generality, the definition adopted here omits to distinguish the marked states.

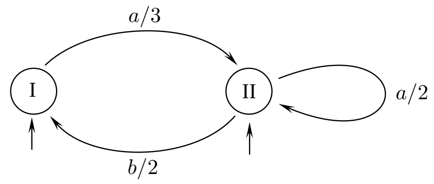

Fig. 1. A non deterministic (max,+) automaton.

We define $x_G(w) \in \mathbb{R}_{\max}^{1\times|Q|}$ by

$$x_G(w) = \alpha\mu(w).$$

An element $[x_G(w)]_q$ is interpreted as the date at which the state q is reached at the end of the events sequence w from an initial state (with the convention that $[x_G(w)]_q = \varepsilon$ if the state q is not reached when w is completed). The elements of x_G are *generalized daters*, and we have

$$\begin{cases} x_G(\epsilon) = \alpha, \\ x_G(wa) = x_G(w)\mu(a). \end{cases} \quad (1)$$

3. NEW REPRESENTATIONS FOR (MAX,+) AUTOMATA

Let us first introduce several notations for a (max,+) automaton $G = (Q, \Sigma, \alpha, \mu)$.We define the set of triples $H \subset Q \times \Sigma \times Q$ as follows:

$$H = \{(q, a, q') \in Q \times \Sigma \times Q \mid [\mu(a)]_{qq'} \neq \varepsilon\}. \quad (2)$$

A triple (q, a, q') belongs to H if there exists a state transition according to event a from state q to state q'.
For a given event $a \in \Sigma$ and state $q \in Q$, we define the set $H_{a,q} \subset H$ by:

$$H_{a,q} = \{(r, \alpha, s) \in H \mid \alpha = a, s = q\}.$$

We also define the set:

$$\sigma_{n,a,q} = \{[x_G(wa)]_q \mid |w| = n - 1\}.$$

Set $\sigma_{n,a,q}$ contains the completion dates for sequences of length n, starting from an initial state, ending with event a and leading to state q.
Let T denote the set of possible durations associated with events, that is,

$$T = \{\tau \mid \exists a \in \Sigma, \exists p \in Q, \exists q \in Q \mid [\mu(a)]_{pq} = \tau\}. \quad (3)$$

Let $\gamma_{t,a,q}$ be the set defined by:

$$\gamma_{t,a,q} = \{|wa| \mid [x_G(wa)]_q \leq t\},$$

This set contains the lengths of sequences completed before or at instant t, ending with event a and leading to state q. Note that sets $\sigma_{n,a,q}$ and $\gamma_{t,a,q}$ are subsets of $\mathbb{R}_{\max}$ and are *chains* (that is totally ordered sets). The representations presented below allow us to determine in particular:

- the maximum element of $\sigma_{n,a,q}$, that is the *maximum execution time* for sequences of length n, ending with a and leading to q;
- a minorant of $\sigma_{n,a,q}$, that is a minorant of the *execution time* for sequences of length n, ending with a and leading to q;
- a majorant of $\gamma_{t,a,q}$, that is a majorant of lengths for sequences completed before or at instant t, ending with a and leading to q.

3.1 Representation corresponding to the worst-case behavior

Let us define matrix $\overline{A} \in \mathbb{R}_{\max}^{|H|\times|H|}$ as follows:
for $j = (p, a, q) \in H$ and $k = (r, a', s) \in H$

$$\overline{A}_{jk} = \begin{cases} [\mu(a)]_{pq} & \text{if } s = p, \\ \varepsilon & \text{otherwise.} \end{cases} \qquad (4)$$

Example 3. The (max,+) automaton represented in figure 1 is such that

$$H = \{(I, a, II), (II, a, II), (II, b, I)\},$$

and

$$\overline{A} = \begin{pmatrix} \varepsilon & \varepsilon & 3 \\ 2 & 2 & \varepsilon \\ 2 & 2 & \varepsilon \end{pmatrix}.$$

For example, triples (I, a, II) and (II, b, I) are listed respectively as 1st and 3rd elements in H and $\overline{A}_{1,3} = 3$ brings the information that state transition (I, a, II) can occur consecutively to the occurrence of state transition (II, b, I) with a duration of 3 time units.

Proposition 1. Let $\overline{x}(n) \in \mathbb{R}_{\max}^{|H|\times 1}$, for $n \in \mathbb{N}$, be defined iteratively by

- for each $j = (p, a, q) \in H$,
$$[\overline{x}(1)]_j = \begin{cases} [\mu(a)]_{pq} & \text{if } p \text{ is an initial state,} \\ \varepsilon & \text{otherwise,} \end{cases} \qquad (5)$$
- for $n > 1$
$$\overline{x}(n) = \overline{A} \otimes \overline{x}(n-1). \qquad (6)$$

Then $\bigoplus_{j\in H_{a,q}} [\overline{x}(n)]_j$ is the maximum element of $\sigma_{n,a,q}$ for each $a \in \Sigma$ and $q \in Q$.

Proof 1. We use mathematical induction to prove the result. Using (5), we have

$$\bigoplus_{j\in H_{a,q}} [\overline{x}(1)]_j = \bigoplus_{\{p\in Q | p \text{ initial state}\}} [\mu(a)]_{pq},$$

which corresponds to the completion date to reach the state q upon the only occurrence of a (completion date for the sequences of length 1, composed of a and leading to q).

Let us assume that $\bigoplus_{j\in H_{\alpha,p}} [\overline{x}(n)]_j$ is the maximum element of $\sigma_{n,\alpha,p}$ for all $\alpha \in \Sigma, p \in Q$.
Let us show that for all $a \in \Sigma, q \in Q$, $\bigoplus_{j\in H_{a,q}} [\overline{x}(n+1)]_j$ is the maximum element of $\sigma_{n+1,a,q}$. We have:

$$\begin{aligned} &\bigoplus_{j\in H_{a,q}} [\overline{x}(n+1)]_j \\ &= \bigoplus_{j\in H_{a,q}} [\overline{A} \otimes \overline{x}(n)]_j \quad \text{(using (6))}, \\ &= \bigoplus_{j\in H_{a,q}} \bigoplus_{l\in H} [\overline{A}]_{jl} \otimes [\overline{x}(n)]_l, \\ &= \bigoplus_{l\in H} \bigoplus_{j\in H_{a,q}} [\overline{A}]_{jl} \otimes [\overline{x}(n)]_l, \\ &= \bigoplus_{p\in Q} \bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} \bigoplus_{j\in H_{a,q}} [\overline{A}]_{jk} \otimes [\overline{x}(n)]_k. \end{aligned}$$

Notice that, by definition (4), we have, for all $k \in H_{\alpha,p}$,

$$\bigoplus_{j\in H_{a,q}} [\overline{A}]_{jk} = \begin{cases} \bigoplus_{p'\in Q} [\mu(a)]_{p'q} & \text{if } p = p', \\ \varepsilon & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} &\bigoplus_{j\in H_{a,q}} [\overline{x}(n+1)]_j, \\ &= \bigoplus_{p\in Q} \bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} [\mu(a)]_{pq} \otimes [\overline{x}(n)]_k, \\ &= \bigoplus_{p\in Q} [\mu(a)]_{pq} \otimes \Big[\bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} [\overline{x}(n)]_k \Big]. \end{aligned}$$

It is assumed that $\bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} [\overline{x}(n)]_k$ represents the maximum completion date for sequences of length n leading to state p, then $\bigoplus_{j\in H_{a,q}} [\overline{x}(n+1)]_j$ is the maximum completion date for sequences of length $n+1$, ending with event a and leading to state q.

Example 4. Let us consider the non deterministic (max,+) automaton represented in figure 1. We have $Q = \{I, II\}$, $\Sigma = \{a, b\}$, $H = \{(I, a, II), (II, a, II), (II, b, I)\}$, and

$$\overline{x}(n) = \begin{bmatrix} \overline{x}_{I,a,II}(n) \\ \overline{x}_{II,a,II}(n) \\ \overline{x}_{II,b,I}(n) \end{bmatrix}, \overline{x}(1) = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}.$$

Vector $\overline{x}(n)$ satisfies the recursive equation (6), that is:

$$\begin{bmatrix} \overline{x}_{I,a,II}(n) \\ \overline{x}_{II,a,II}(n) \\ \overline{x}_{II,b,I}(n) \end{bmatrix} = \begin{pmatrix} \varepsilon & \varepsilon & 3 \\ 2 & 2 & \varepsilon \\ 2 & 2 & \varepsilon \end{pmatrix} \otimes \begin{bmatrix} \overline{x}_{I,a,II}(n-1) \\ \overline{x}_{II,a,II}(n-1) \\ \overline{x}_{II,b,I}(n-1) \end{bmatrix}.$$

The following table contains the first values obtained thanks to this recurrence in $\mathbb{R}_{\max}$.

Table 1.

n	1	2	3	4	5	...
$\overline{x}_{I,a,II}(n)$	3	5	8	10	13	...
$\overline{x}_{II,a,II}(n)$	2	5	7	10	12	...
$\overline{x}_{II,b,I}(n)$	2	5	7	10	12	...

For example, the possible sequences of length 3 ending with event a and leading to state II are $\{aaa, aba, baa\}$. We obtain using (1)

$$[x_G(aaa)]_{II} = 7, \ [x_G(aba)]_{II} = 8, \ [x_G(baa)]_{II} = 7,$$

which leads to $\sigma_{3,a,II} = \{7, 8\}$.
On the other hand, we have

$$H_{a,II} = \{(I, a, II), (II, a, II)\},$$

hence

$$\bigoplus_{j\in H_{a,II}} [\overline{x}(3)]_j = \overline{x}_{I,a,II}(3) \oplus \overline{x}_{II,a,II}(3) = 8,$$

which corresponds to the maximum element of $\sigma_{3,a,II}$, that is the maximum completion time for sequences of length 3 ending by event a and leading to state II.

3.2 Representation approximating the optimal case behavior for sequences durations

In this section, we define a representation in a very similar way to the previous-section one, but over (min,+) algebra instead of (max,+) algebra. The reader must then have in mind that $\oplus$ represents the min operation and $\varepsilon = +\infty$.

Matrix $\underline{A} \in \mathbb{R}_{\min}^{|H|\times|H|}$ is defined by
for $j = (p, a, q) \in H$, $k = (r, a', s) \in H$,

$$\underline{A}_{jk} = \begin{cases} [\mu(a)]_{pq} & \text{if } s = p, \\ \varepsilon & \text{otherwise.} \end{cases} \quad (7)$$

Proposition 2. Let $\underline{x}(n) \in \mathbb{R}_{\min}^{|H|\times 1}$, for $n \in \mathbb{N}$, be defined iteratively by
for each $j = (p, a, q) \in H$,

$$[\underline{x}(1)]_j = \begin{cases} [\mu(a)]_{pq} & \text{if } p \text{ is an initial state,} \\ \varepsilon & \text{otherwise.} \end{cases} \quad (8)$$

$$\underline{x}(n) = \underline{A} \otimes \underline{x}(n-1), \text{ for } n > 1. \quad (9)$$

For all $a \in \Sigma, q \in Q$, $\bigoplus_{j\in H_{a,q}} [\underline{x}(n)]_j$ is a minorant for $\sigma_{n,a,q}$, that is a minorant of the possible completion dates of sequences of length n ending by event a and leading to state q.

Proof 2. The proof goes along the same lines as the proof of proposition 1 and also proceeds by induction. According to (8), we have

$$\begin{aligned} \bigoplus_{j\in H_{a,q}} [\underline{x}(1)]_j &= \bigoplus_{\{p\in Q|p \text{ initial state}\}} [\mu(a)]_{pq} \\ &= \min_{\{p\in Q|p \text{ initial state}\}} [\mu(a)]_{pq} \end{aligned}$$

The completion date leading to the state q upon the occurrence of the only event a is given by $\alpha\mu(a)]_q$, that is, $\max_{p \text{ initial state}}([\mu(a)]_{pq})$. It should be clear that $\bigoplus_{j\in H_{a,q}} [\underline{x}(1)]_j$ is then lower (possibly strictly) than $\sigma_{1,a,q}$.
Assuming that for all $\alpha \in \Sigma, p \in Q$, $\bigoplus_{j\in H_{\alpha,p}} [\underline{x}(n)]_j$ is a minorant of $\sigma_{n,\alpha,p}$, the same arguments as those used in the proof of proposition 1 lead to

$$\begin{aligned} \bigoplus_{j\in H_{a,q}} [\underline{x}(n+1)]_j &= \bigoplus_{p\in Q} [\mu(a)]_{pq} \otimes \left[\bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} [\underline{x}(n)]_k \right] \\ &= \min_{p\in Q} \left\{ [\mu(a)]_{pq} \otimes \left[\bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} [\underline{x}(n)]_k \right] \right\}. \end{aligned}$$

Since $\bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} [\underline{x}(n)]_k$ is a minorant for the completion dates of sequences of length n leading to state p, we can claim that $\bigoplus_{j\in H_{a,q}} [\underline{x}(n+1)]_j$ is also a minorant for the completion dates of sequences of length $n+1$ ending by event a and leading to state q.

Corollary 1. If G is a deterministic (max,+) automaton, then $\bigoplus_{j\in H_{a,q}} [\underline{x}(n)]_j$ is the minimum element of $\sigma_{n,a,q}$.

Example 5. Let us consider again the (max,+) automaton represented in figure 1. Table 5 contains the first values obtained for $\underline{x}(n)$ thanks to recurrence (9) in $\mathbb{R}_{min}$.

Table 2.

n	1	2	3	4	5	...
$\underline{x}_{I,a,II}(n)$	3	5	7	9	11	...
$\underline{x}_{II,a,II}(n)$	2	4	6	8	10	...
$\underline{x}_{II,b,I}(n)$	2	4	6	8	10	...

We have mentioned in the last example that the set of the possible completion dates for the sequences of length 3 ending by event a and leading to state II, is given by

$$\sigma_{3,a,II} = \{7, 8\},$$

and $H_{a,II} = \{(I, a, II), (II, a, II)\}$. Then we have

$$\begin{aligned} \bigoplus_{j\in H_{a,II}} [\underline{x}(3)]_j &= \underline{x}_{I,a,II}(3) \oplus \underline{x}_{II,a,II}(3) \\ &= \min(\underline{x}_{I,a,II}(3), \underline{x}_{II,a,II}(3)) \\ &= 6 \end{aligned}$$

For this example, $\bigoplus_{j\in H_{a,II}} [\underline{x}(3)]_j$ is a minorant of set $\sigma_{3,a,II}$.

3.3 Representation approximating the optimal-case behavior for sequence lengths

As in section 3.2, the following representation enables us to study the optimal case behavior of the automaton but for evaluating the greatest sequence-length until a given date. This representation is defined over $\mathbb{R}_{\max}$.
We define the matrices denoted $\overline{E}_\tau \in \mathbb{R}_{\max}^{|H|\times|H|}$ as follows: for all $\tau \in T$, $j = (p, a, q) \in H$ and $k = (r, a', s) \in H$

$$[\overline{E}_\tau]_{jk} = \begin{cases} 1 & \text{if } s = p \text{ and } [\mu(a)]_{pq} = \tau, \\ \varepsilon & \text{otherwise.} \end{cases} \quad (10)$$

Proposition 3. Let $\overline{z}(t) \in \mathbb{R}_{\max}^{|H|\times 1}$, $t \in \mathbb{N}$, be defined iteratively as follows with $\tau_{\min} = \min(T)$ and $\tau_{\max} = \max(T)$:

- for $t < \tau_{\min}$:
  $$[\overline{z}(t)]_j = \varepsilon \qquad \forall j \in H,$$
- for $\tau_{min} \le t \le \tau_{max}$: $[\overline{z}(t)]_j$ is set to be equal to the maximum length of sequences going from initial state to state q, ending with event a before or at date t,
- for $t > \tau_{max}$:
  $$\overline{z}(t) = \bigoplus_{\tau\in T} \overline{E}_\tau \otimes \overline{z}(t-\tau) \oplus \overline{z}(t-1). \quad (11)$$

Then $\bigoplus_{j\in H_{a,q}} [\overline{z}(t)]_j$ is a majorant of set $\gamma_{t,a,q}$ for each $a \in \Sigma$ and $q \in Q$, that is a majorant of the possible lengths among sequences ending by event a and leading to state q before or at date t.

Proof 3. Once more, we use mathematical induction.
For $t \le \tau_{max}$ the result of the proposition 3 is true by definition. Let us assume that $\bigoplus_{k\in H_{\alpha,p}} [\overline{z}(t)]_k$ is a majorant of set $\gamma_{t,\alpha,p}$ for all $\alpha \in \Sigma, p \in Q$, and we show that for all $a \in \Sigma$, $q \in Q$, $\bigoplus_{j\in H_{a,q}} [\overline{z}(t+\tau_{\min})]_j$ is a majorant of set $\gamma_{t+\tau_{\min},a,q}$.
We have

$$\begin{aligned} \overline{z}(t+\tau_{\min}) &= \bigoplus_{\tau\in T} \overline{E}_\tau \otimes \overline{z}(t+\tau_{\min}-\tau) \oplus \overline{z}(t+\tau_{\min}-1) \\ &\ge \overline{E}_{\tau_{\min}} \otimes \overline{z}(t) \oplus \overline{z}(t+\tau_{\min}-1) \\ &\ge \overline{E}_{\tau_{\min}} \otimes \overline{z}(t). \end{aligned}$$

Hence,

$$\begin{aligned} \bigoplus_{j\in H_{a,q}} [\overline{z}(t+\tau_{\min})]_j &\ge \bigoplus_{j\in H_{a,q}} [\overline{E}_{\tau_{\min}} \otimes \overline{z}(t)]_j \\ &= \bigoplus_{j\in H_{a,q}} \bigoplus_{l\in H} [\overline{E}_{\tau_{\min}}]_{jl} \otimes \overline{z}(t)]_l. \end{aligned}$$

By definition, we have for $j = (p, a, q)$

$$[\overline{E}_{\tau_{\min}}]_{jl} \otimes \overline{z}(t)]_l = 1 \otimes \overline{z}(t)]_l$$

for $l = (r, a', s)$ such that $s = p$ and $[\mu(a)]_{pq} = \tau_{\min}$.
This leads to

$$\bigoplus_{j\in H_{a,q}} [\overline{z}(t+\tau_{\min})]_j \geq 1 \otimes \left[\bigoplus_{\{p\in Q|(p,a,q)\in H\}} \bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} \overline{z}(t)]_k \right],$$

which shows that $\bigoplus_{j\in H_{a,q}} [\overline{z}(t+\tau_{\min})]_j$ is a majorant of set $\gamma_{t+\tau_{\min},a,q}$.

Example 6. We consider the (max,+) automaton represented in figure 1.

$$\overline{z}(t) = \begin{bmatrix} \overline{z}_{I,a,II}(t) \\ \overline{z}_{II,a,II}(t) \\ \overline{z}_{II,b,I}(t) \end{bmatrix},$$

$$\overline{E}_2 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ 1 & 1 & \varepsilon \\ 1 & 1 & \varepsilon \end{pmatrix}, \overline{E}_3 = \begin{pmatrix} \varepsilon & \varepsilon & 1 \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

Initial values of vector $\overline{z}(t)$ are:

$$\overline{z}(1) = \begin{bmatrix} \varepsilon \\ \varepsilon \\ \varepsilon \end{bmatrix}, \overline{z}(2) = \begin{bmatrix} 1 \\ \varepsilon \\ 1 \end{bmatrix}, \overline{z}(3) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For $t > 3$,vector $\overline{z}(t)$ satisfies the recursive equation (11), that is:

$$\overline{z}(t) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ 1 & 1 & \varepsilon \\ 1 & 1 & \varepsilon \end{pmatrix} \otimes \overline{z}(t-2) \oplus \begin{pmatrix} \varepsilon & \varepsilon & 1 \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \otimes \overline{z}(t-3) \oplus \overline{z}(t-1)$$

The following table contains the first values obtained thanks to this recurrence in $\mathbb{R}_{\max}$.

Table 3.

t	1	2	3	4	5	6	7	...
$\overline{z}_{I,a,II}(t)$	ε	1	1	1	2	3	3	...
$\overline{z}_{II,a,II}(t)$	ε	ε	1	2	2	2	3	...
$\overline{z}_{II,b,I}(t)$	ε	1	1	2	2	2	3	...

For example, the possible sequences ending with event a and leading to state II, executed at or before instant $t = 6$ are: $\{a, ba, aa\}$. Note that $[x_G(aaa)]_{II} = 7$ and therefore string aaa does not belong to this set. We have $\gamma_{6,a,II} = \{1, 2\}$.
On the other hand, we have

$$H_{a,II} = \{(I, a, II), (II, a, II)\},$$

hence

$$\bigoplus_{j\in H_{a,II}} [\overline{z}(6)]_j = \overline{z}_{I,a,II}(6) \oplus \overline{z}_{II,a,II}(6) = 2,$$

which corresponds to the maximum element of $\gamma_{6,a,II}$, that is the maximum length sequence ending by event a, leading to state II and completed before or at instant $t = 6$.

4. APPLICATIONS TO PERFORMANCE EVALUATION

Let us now focus on applications of the new representations for performance evaluation and discuss related results in the literature Gaubert (1995), Su and Woeginger (2011), Gaubert and Mairesse (1999).

4.1 Maximum execution time

For some systems, it is important to have knowledge of the maximum execution time for sequences of given length n, that is the maximum element of the set composed of completion times for sequences corresponding to n events. Its calculation is presented in Gaubert (1995) as follows:

$$\begin{aligned} l_n^{worst} &= \bigoplus_{w\in\Sigma^n} \bigoplus_{p\in Q} [x_G(w)]_p, \\ &= \bigoplus_{w\in\Sigma^n} \bigoplus_{p\in Q} [\alpha\mu(a_1)...\mu(a_n)], \\ &= \bigoplus_{p\in Q} [\alpha M^n]_p, \end{aligned}$$

with $M = \bigoplus_{a\in\Sigma} \mu(a)$.

The time complexity of the computation above is $O(n^4|\mathcal{R}|^3)$, where $\mathcal{R}$ is a clique covering of Σ.
Another computation method, using *heap models*, is presented in Su and Woeginger (2011) with a better time complexity, that is $O(|H||\mathcal{R}|^2)$, where H (defined by (3)), corresponds to set of edges in the (max,+) automaton G. Nevertheless, it must be noticed that the class of (max,+) automata considered in Su and Woeginger (2011) is more restrictive that the one studied in Gaubert (1995) and in this paper, since it is assumed that for each event one, and only one, possible time duration is associated.

The representation introduced in proposition 1 also enables us to evaluate this indicator since

$$l_n^{worst} = \bigoplus_{j\in H} [\overline{x}(n)]_j = \bigoplus_{j\in H} [\overline{A}^{n-1} \overline{x}(1)]_j.$$

The time complexity is the one of n multiplications in $\mathbb{R}_{\max}^{|H|\times|H|}$, that is $O(n|H|^3)$.
The comparison with the methods mentioned previously doesn't give always the same conclusion. In fact, as noticed in Hall and Jr. (1941), the maximal cardinality of $|\mathcal{R}|$ is $\frac{|\Sigma|^2}{4}$ and there exist automata in which $|\Sigma|^2 < |H|$ as well as others in which $|\Sigma|^2 \geq |H|$.

Remark 1. As in Gaubert (1995), our approach can take profit of the spectral properties of matrix in $\mathbb{R}_{\max}$ to simplify the computation.
For an irreducible matrix $\overline{A} \in \mathbb{R}_{\max}^{|H|\times|H|}$ with as eigenvalue $\lambda = \bigoplus_{k=1}^{|H|} (tr\overline{A}^k)^{\frac{1}{k}}$, we have $\overline{A}^{n+c} = \lambda^c \overline{A}^n$. This allows us to claim that if the value of $\overline{A}^n$ (resp. l_n^{worst}) is already computed, the computation of $\overline{A}^{n+c}$ (resp. l_{n+c}^{worst}), $\forall n \geq N$ reduces to operation $\lambda^c \overline{A}^n$.

4.2 Minimum execution time

For other systems , it is important to be able to compute the minimum execution time for sequences of given length n. It is referred to as the *optimal case* in Gaubert (1995) and formulated as follows:

$$l_n^{opt} = \bigoplus_{w\in\Sigma^n} \bigoplus_{p\in Q} [x_G(w)]_p,$$

where $\bigoplus$ represents the min operator.
The algorithm which is given in Gaubert (1995) only applies for a reduced class of (max,+) automata (deterministic automata) and is announced to be much more onerous than for the worst-case. In Su and Woeginger (2011), it is shown that the algorithm of computation of the "optimal-case" is NP-complete.
In proposition 2, based on the new representation 8-9, a minorant of this element is obtained by a simple recurrence in $\mathbb{R}_{min}$, as:

$$l_n^{opt} \geq \bigoplus_{j \in H} [\underline{x}(n)]_j = \bigoplus_{j \in H} [\underline{A}^{n-1} \underline{x}(1)]_j.$$

The time complexity of this computation is the same as that of the "maximum execution time", that is $\mathcal{O}(n|H|^3)$. Note that spectral properties of matrices over $\mathbb{R}_{min}$ can also be exploited to simplify the calculus.

4.3 Maximum length of sequences for a given executing time

The representation proposed in section 3.3 allows us to compute another performance indicator which, to the best of our knowledge, has not been discussed in the literature, that is:

$$\bigoplus_{j \in H_{a,q}} [\overline{z}(t)]_j.$$

This indicator gives information about the optimal behavior of the automaton with a focus on the greatest number of events occurring until a given time instant. The computation of this indicator has a similar time complexity than the indicators presented previously since it merely implies matrix multiplication in $\mathbb{R}_{\max}$.

5. CONCLUSION

We have proposed new representations for (max,+) automata in order to describe their extremal behaviors. We have shown that these representations can be applied to performance evaluation. In particular, maximum and minimum execution times for sequences of given length, as well as the maximum number of events occurring until a given time instant can be evaluated or at least bounded.
A control approach, inspired by that for logical automata presented in Ramadge and Wonham (1989), has been proposed in Komenda et al. (2009b). It is then assumed that automata are deterministic to guarantee realizability of controllers (see the discussion of the first paragraph in introduction). The representations presented in this contributions could be used to elaborate alternative control laws for general automata. Since these representations are similar to standard state-space ones, it is envisaged to transpose the control laws developed for linear (max,+) and (min,+) systems (see for example Lahaye et al. (1999), Houssin et al. (2007) and Amari et al. (2012)).

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