Control of $(max, +)$-linear systems minimizing delays

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Abstract

In this paper, we develop a new control technique for discrete event dynamic systems subject to synchronization phenomena. We propose a feedback controller for $(max, +)$-linear systems which delays input events as less as possible while constraints on internal or output events are satisfied. The synthesis is mainly based on new results about fixed points of antitone (*i.e.*, order reversing) mappings.

Index Terms

Discrete event systems; $(max, +)$-linear systems; controller synthesis.

I. INTRODUCTION

In this paper, we study Discrete Events Dynamic Systems (DEDS) that can be modeled by a linear representation in $(max, +)$ algebra. This class of DEDS corresponds to Timed Event Graphs (TEG). A linear system theory has been developed for these particular systems in [1] with applications to flexible manufacturing systems, telecommunication and transportation networks [8]. Strong analogies exist between the classical linear system theory and the $(max, +)$-linear system theory. In particular, the concept of control is well defined in context of TEG. It refers to the firing control of TEG input transitions in order to reach a desired performance.

One possible approach is based on the *model-reference technique*. A given model then describes the desired performance limits and the design goal is achieved through the calculation of

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a precompensator and/or of a feedback controller. Such a controller is computed *a priori* and is valid for all potential reference inputs. In these works, such as [5] and [12], the authors consider a just-in-time criterion, that is, the proposed control laws delay occurrences of input or internal events (firings of input or internal transitions) as much as possible. This paper also deals with the *a priori* synthesis of a feedback controller valid for all possible reference inputs. However, the control objective as well as the technique are different and, to our knowledge, original. Instead of the just-in-time criterion, the aim of the control is to delay the system as less as possible, that is to postpone the occurrences of input events (firings of input transitions) as less as possible, while satisfying some given constraints (rather than a model-matching problem). For example, in a railway network, one can aim at limiting the number of trains on a path (by increasing dwell times at stations to improve connections) while minimizing the induced delays. Another possible application concerns push flow production systems subject to critical time constraints, in which sojourn times of pieces must not exceed a given value at some stages. Hence, we may be interested at bounding the sojourn times while delaying the release of raw parts into the system as less as possible. For such control problems, our approach is mainly based on new results about fixed points of antitone (order reversing) mappings.

For the control of TEG, several techniques of *reference signal tracking* have also been studied. In particular, an extension of Model Predictive Control (MPC) has been proposed for $(max, +)$-linear systems in [13], and notably applied to the just-in-time output tracking problem: compute the latest occurrence times of consecutive inputs events while minimizing the error between a reference signal (defining due dates for the output events) and the predicted output of the system. An advantage of this approach is that it can accommodate constraints on the inputs and outputs. It will be pointed out that the criterion and the constraints considered in the present paper can be recast in the MPC framework. Nevertheless, the obtained control law then depends on the reference input whereas the proposed feedback is valid for all possible reference inputs.

In section 2, we recall some results from the dioid theory and introduce results concerning isotone and antitone mappings. Section 3 is devoted to the modeling of DEDS. The proposed control laws are presented in section 4 before to conclude.

II. ALGEBRAIC TOOLS

A. Dioid theory

A dioid $(\mathcal{D}, \oplus, \otimes)$ is a semi-ring in which the sum, denoted by $\oplus$, is idempotent. The sum (resp. product) admits a neutral element denoted ε (resp. e). A dioid is said to be complete if it is closed for infinite sums and if the product distributes over infinite sums too. The sum of all its elements is generally denoted $\top$ (for top).

Example 1: The set $\mathbb{Z}_{max} = \mathbb{Z} \cup \{-\infty\}$ endowed with the max operator as sum and the classical sum as product is a (non-complete) dioid. If we add $\top = +\infty$ (with the convention $\top \otimes \varepsilon = +\infty + (-\infty) = -\infty = \varepsilon$) to this set, the resulting dioid is complete and is denoted $\overline{\mathbb{Z}}_{max}$.

Due to the idempotency of the sum, a dioid is endowed with a partial order relation, denoted $\succeq$ and defined by the following equivalence: $a \succeq b \Leftrightarrow a = a \oplus b$. A complete dioid has a structure of complete lattice [1, §4], *i.e.*, two elements in a complete dioid always have a *least upper bound*, namely $a \oplus b$, and a *greatest lower bound* denoted $a \wedge b = \bigoplus_{\{x | x \preceq a, \ x \preceq b\}} x$ in the considered dioid.

Let $\mathcal{D}$ and $\mathcal{C}$ be two complete dioids. A mapping $f : \mathcal{D} \to \mathcal{C}$ is said to be isotone (resp. antitone) if $a, b \in \mathcal{D}$, $a \preceq b \Rightarrow f(a) \preceq f(b)$ (resp. $f(a) \succeq f(b)$).

Residuation theory [3] defines "pseudo-inverses" for some isotone mappings defined over ordered sets such as complete dioids [4]. In particular, if the greatest element of the set $\{x \in \mathcal{D} | f(x) \succeq b\}$ exists for all $b \in \mathcal{C}$, then it is denoted $f^\flat(b)$ and $f^\flat$ is called *dual residual* of f.

Example 2: The mapping $T_a : \mathcal{D} \to \mathcal{D}; x \mapsto a \oplus x$ is dually residuated (see [1, §4.4.4] for a proof). The dual residual is denoted $T_a^\flat(b) = b \ominus a$. It should be clear that $a \succeq b \Leftrightarrow T_a^\flat(b) = \varepsilon$. If T_a is defined over $\overline{\mathbb{Z}}_{max}$ then

$$T_a^\flat(b) = b \ominus a = \begin{cases} b & \text{if } b > a, \\ \varepsilon & \text{otherwise.} \end{cases}$$

We recall the following property of $T_a^\flat$ used later:

$$a(x \ominus b) \succeq ax \ominus ab. \tag{1}$$

Note that the product operator $\otimes$ has been (and will be) omitted as soon as no ambiguity can appear. A relevant remark is that although $T_a^\flat(x) = x \ominus a$ is isotone, the mapping $x \mapsto a \ominus x$

is antitone since $x_1 \preceq x_2 \Leftrightarrow a \oslash x_1 \succeq a \oslash x_2, \forall a$. It should be clear that $a \oslash x_1$ is the least solution of $x_1 \oplus x \succeq a$ and $a \oslash x_2$ is the least solution of $x_2 \oplus x \succeq a$ (see [1, 4.4.4] for more details).

B. Fixed points of mappings defined over dioids

Because of their lattice structure, properties about fixed points stated for lattices also apply over dioids.

Notation 1: Let $f : \mathcal{D} \rightarrow \mathcal{D}$ with $\mathcal{D}$ a complete dioid, we use the following notations: $\mathcal{F}_f = \{x \in \mathcal{D} | f(x) = x\}$, $\mathcal{P}_f = \{x \in \mathcal{D} | f(x) \succeq x\}$, $\mathcal{Q}_f = \{x \in \mathcal{D} | f(x) \preceq x\}$ and f^2 denotes $f \circ f$.

For an isotone mapping f, in [14] and [6] it has been shown that the sets $\mathcal{F}_f$, $\mathcal{P}_f$ and $\mathcal{Q}_f$ are non-empty complete lattices. Moreover, it can be shown that the greatest (resp. least) fixed point coincides with the greatest (resp. least) element of $\mathcal{P}_f$ (resp. $\mathcal{Q}_f$):

$$
\begin{aligned}
Sup \ \mathcal{P}_f = Sup \ \mathcal{F}_f \quad &\text{and} \quad Sup \ \mathcal{F}_f \in \mathcal{F}_f, \\
Inf \ \mathcal{Q}_f = Inf \ \mathcal{F}_f \quad &\text{and} \quad Inf \ \mathcal{F}_f \in \mathcal{F}_f.
\end{aligned}
\tag{2}
$$

In the following proposition given without proof, a well known method to compute the greatest fixed point of an isotone mapping f (see for example [15]) is indicated.

Proposition 1: Let f be an isotone mapping. If the following iterative computation

$$
\begin{aligned}
y_0 &= \top \\
y_{k+1} &= f(y_k)
\end{aligned}
\tag{3}
$$

converges in a finite number k_e of iterations, then y_{k_e} is the greatest fixed point of f.

Properties about fixed points of antitone mappings are not well established, and only few works have tackled this problem [2], [7]. To the best of our knowledge, results presented in the sequel are original. However, proposition 6 has been inspired by [7, th. A]. More details and illustrations can be found in [11].

Notice that if f is an antitone mapping then f^2 is isotone. Let us first characterize the structure of $\mathcal{P}_f$ and $\mathcal{Q}_f$.

Proposition 2: Let $f : \mathcal{D} \rightarrow \mathcal{D}$ be an antitone mapping. The set $\mathcal{Q}_f$ (resp. $\mathcal{P}_f$) is a complete upper semi-lattice (resp. complete lower semi-lattice).

Proof: Let us consider two elements $x, y \in \mathcal{Q}_f$. Since f is antitone, we have $f(x \oplus y) \preceq f(x)$ and $f(x \oplus y) \preceq f(y)$ which implies that $f(x \oplus y) \preceq f(x) \wedge f(y) \preceq f(x) \oplus f(y) \preceq x \oplus y$, and hence $x \oplus y \in \mathcal{Q}_f$. This assertion also applies to infinite sums. The set $\mathcal{P}_f$ is proved to be a complete lower semi-lattice by identical arguments. ∎

Proposition 3: Let $f : \mathcal{D} \to \mathcal{D}$ be an antitone mapping and $x \in \mathcal{D}$. We have

$$x \oplus f(x) \;\; \in \;\; \mathcal{Q}_f, \quad x \wedge f(x) \;\; \in \;\; \mathcal{P}_f.$$

Proof: We have $f(x) \oplus x \succeq x$ and $f(x) \oplus x \succeq f(x)$ which implies by antitony of f that $f(f(x) \oplus x) \preceq f(x) \preceq f(x) \oplus x$. Similarly, $f(x) \wedge x \in \mathcal{P}_f$ since $f(f(x) \wedge x) \succeq f(x) \succeq f(x) \wedge x$. ∎

Proposition 4: Let $f : \mathcal{D} \to \mathcal{D}$ be an antitone mapping, $y \in \mathcal{P}_f$ and $z \in \mathcal{Q}_f$. For all $x \in \mathcal{D}$ such that $x \preceq y$ (resp. $x' \in \mathcal{D}$ such that $x' \succeq z$), we have $x \in \mathcal{P}_f$ (resp. $x' \in \mathcal{Q}_f$).

Proof: We use the antitony of f :

$$
\begin{aligned}
x &\preceq y \Rightarrow f(x) \succeq f(y) \succeq y \succeq x, \\
x' &\succeq z \Rightarrow f(x') \preceq f(z) \preceq z \preceq x'.
\end{aligned}
$$

∎

Proposition 5: If x is a fixed point of an antitone mapping $f : \mathcal{D} \to \mathcal{D}$, then x is a minimal (resp. maximal) element of $\mathcal{Q}_f$ (resp. $\mathcal{P}_f$).

Proof: Let $x \in \mathcal{F}_f$, $y \in \mathcal{P}_f$ and $z \in \mathcal{Q}_f$ such that $y \succeq x \succeq z$. Using the antitony of f, we obtain $f(y) \preceq f(x) \preceq f(z) \Rightarrow y \preceq f(y) \preceq x \preceq f(z) \preceq z$, and hence $y = x = z$. We conclude that there is no element of $\mathcal{Q}_f$ (resp. $\mathcal{P}_f$) which is less (resp. greater) than x. ∎

As a corollary to this proposition, notice that if f admits several distinct fixed points, then they are not comparable. Furthermore, remark that set $\mathcal{F}_f$ can be empty.

Proposition 6: Let $f : \mathcal{D} \to \mathcal{D}$ be an antitone mapping. Denoting $\mu = Inf \; \mathcal{F}_{f^2}$ and $\nu = Sup \; \mathcal{F}_{f^2}$, we have $\mu \in \mathcal{P}_f$ and $\nu \in \mathcal{Q}_f$.

Proof: We show that $f(\mu) = \nu$ and $f(\nu) = \mu$ (since $\mu \preceq \nu$ by definition, this proves $f(\mu) \succeq \mu$ and $f(\nu) \preceq \nu$). Inserting $\mu = Inf \; \mathcal{F}_{f^2}$ into $f(\cdot)$ yields

$$f(\mu) = f(\bigwedge_{x \in \mathcal{F}_{f^2}} x) \succeq \bigoplus_{x \in \mathcal{F}_{f^2}} f(x) \tag{4}$$

(f antitone $\Rightarrow f(a \wedge b) \succeq f(a) \oplus f(b)$).

However, elements of $\{f(x)|x \in \mathcal{F}_{f^2}\}$ are fixed points of f^2 too since $f^2(f(x)) = f(f^2(x)) = f(x)$. Next we show that $f_{|\mathcal{F}_{f^2}}$ is a permutation. Indeed consider that $x, y \in \mathcal{F}_{f^2}$, $x \neq y$ and $f(x) = f(y)$, then we have $f^2(x) = f^2(y)$ and so $x = y$ which is a contradiction. Hence inequality (4) can be rewritten as

$$ f(\mu) \succeq \bigoplus_{y \in \mathcal{F}_{f^2}} y = \nu. $$

We previously remarked that $f(x)$ with $x \in \mathcal{F}_{f^2}$ is a fixed point of f^2 so does $f(\mu)$ and it leads to $f(\mu) = \bigoplus_{y \in \mathcal{F}_{f^2}} y = \nu$. From the last equality, we obtain also $f(f(\mu)) = \mu = f(\nu)$. $\blacksquare$

As any fixed point of f is also fixed point of f^2, the following corollary follows from Proposition 6.

Corollary 1: If $\nu = \mu$, then $\mathcal{F}_f = \{\nu\}$ and ν is a minimal element of $\mathcal{Q}_f$.

Remark 1: For the following control problem, we are interested in the computation of as small as possible elements of $\mathcal{Q}_f$. The element ν, which can be computed using proposition 1, can be a minimal element of $\mathcal{Q}_f$ (see Corollary 1). Otherwise, it constitutes an interesting upper approximation of a minimal element of $\mathcal{Q}_f$. In fact, any $x \in \mathcal{F}_f$ is a minimal element of $\mathcal{Q}_f$ (see proposition 5) and is such that $x \preceq \nu$ (since x also belongs to $\mathcal{F}_{f^2}$).

III. MODELING DEDS USING DIOIDS

A. State and transfer representation

Dioids enable one to obtain linear models for DEDS which involve (only) synchronization and delay phenomena (but not choice phenomena).

This class of DEDS can be modeled by TEG.

The behavior of such systems can be represented by some discrete functions called *dater* functions (see [1], [8]). More precisely, a discrete variable $x(\cdot)$ is associated to an event labeled x (firing times of transition labeled x in the corresponding TEG). This variable represents the occurring dates of event x. For instance, considering the TEG drawn in figure 1, under the earliest functionning rule[1], the dater x_3 is related to the daters x_1 and x_2 over $\overline{\mathbb{Z}}_{max}$ as follows: $x_3(k) = 1x_1(k) \oplus 2x_2(k-1)$. More generally, every TEG admits a linear *state equation*

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[1]That is, considering that transitions of the TEG are fired as soon as possible.

$$x(k) = Ax(k-1) \oplus Bu(k), \tag{5}$$

where x and u are the state and the input vectors.

An analogous transform to the $\mathcal{Z}$-transform (used to represent discrete-time trajectories in conventional theory) can be introduced for TEG: the γ, δ-transform. This transform enables us to manipulate formal power series, with two commutative variables γ and δ, representing dater trajectories. The set of these formal series is a complete dioid denoted $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ with $e = \gamma^0 \delta^0$ as neutral element of the product and $\varepsilon = (\gamma^{-1})^*(\delta^1)^*$ as neutral element of the sum (the construction of this dioid is detailed in [1]). In the following, we denote by x the corresponding element of $\{x(k)\}_{k \in \mathbb{Z}}$ in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. For instance, the formal series x_3 associated to the corresponding transition in fig. 1 is related to the formal series x_1 and x_2 as follows: $x_3 = \delta^1 x_1 \oplus \gamma^1 \delta^2 x_2$. We can interpret γ as the backward shift operator in event domain and δ as the backward shift operator in time domain. In $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$, the state representation (5) becomes

$$x = Ax \oplus Bu, \tag{6}$$

in which entries of matrices A and B are elements of $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. The least solution is given by $x = A^*Bu$ with $A^* = \bigoplus_{i \in \mathbb{N}} A^i$, $A^0 = e$ and $A^{n+1} = A \otimes A^n$ [1, Th 4.75], and A^*B corresponds to the *transfer* between u and x.

Assumption 1: We assume that the input matrix B is a diagonal square matrix with entries equal to e or ε.

Assumption 1 is not restrictive since it can always be satisfied by extending the state and input vectors and permuting states. We assume that it holds throughout the remainder of the paper. Note that the assumed structure of B is such that $B \preceq e$ and $B^n = B$ for $n \geq 1$.

Example 3: The TEG drawn in fig. 1 can be modeled by (6) with

$$A = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \gamma^1 \delta^1 & \varepsilon & \varepsilon \\ \delta^1 & \gamma^1 \delta^2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon \\ \delta^3 & \varepsilon & \varepsilon & \delta^1 & \varepsilon \end{pmatrix},$$

and, according to assumption 1, B a diagonal matrix s.t. $B_{ii} = e$ if $i \in \{1, 2\}$ and $B_{ii} = \varepsilon$ otherwise.

B. Causality and causal upper approximation

The variables $x \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ used to model TEG satisfy the causality property [1].

Definition 1: Let $x \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$, x is said to be causal if either $x = \varepsilon$ or all exponents of x are in $\mathbb{N}$. A matrix is said *causal* if its entries are all causal. The set of causal elements of $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is a complete dioid denoted $\mathcal{M}_{in}^{ax+}[\![\gamma, \delta]\!]$.

Considering a TEG, a causal transfer means that the system does not require any anticipation (either in time or in event). We now introduce the notion of causal upper approximation (see [10, §2.4]) which will be used later to find a causal controller (*i.e.*, without any anticipation).

Proposition 7: Let $x \in \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$. The two following assertions are equivalent:

(i) x has no negative exponent in γ,

(ii) there exists a least $x' \in \mathcal{M}_{in}^{ax+}[\![\gamma, \delta]\!]$ such that $x' \succeq x$. It means that x admits a causal upper approximation.

Proof: If x is causal, the proof is obvious and $x' = x$. We now consider x not causal. We can limit the proof to the case of monomials since a series is nothing more than a sum of monomial.

(i)⇒(ii) : Let $x = \gamma^n \delta^t$, with $n \geqslant 0$ and $t < 0$. It is easy to see that the monomial $\gamma^n \delta^0$ is the least element of $\mathcal{M}_{in}^{ax+}[\![\gamma, \delta]\!]$ such that $x' \succeq x$. Hence, $x' = \gamma^n \delta^0$.

(ii)⇒(i) : If there exists a least $x' \in \mathcal{M}_{in}^{ax+}[\![\gamma, \delta]\!]$ such that $x' \succeq x$ with $x' = \gamma^{n'} \delta^{t'}$ and $x = \gamma^n \delta^t$, we have $n' \leq n$ and $t' \geq t$. However $x' \in \mathcal{M}_{in}^{ax+}[\![\gamma, \delta]\!]$, hence $n' \geq 0$ and we obtain $n \geq 0$. ∎

We can remark that proposition (7) is also valid for matrices with entries in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$.

We now demonstrate that if an element x admits a causal upper approximation then every element less than x admits a causal approximation too.

Corollary 2: Let x be an element of $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ which admits a causal approximation. Every element y such that $y \preceq x$ admits also a causal approximation.

Proof: The series x can be rewritten as $x = \bigoplus_{i \in I} \gamma^{n_i} \delta^{t_i}$. Since x admits a causal approximation, we have $\forall i \in I$, $n_i \geqslant 0$, in other words, $\min_{i \in I} n_i \geqslant 0$. If $y = \bigoplus_{j \in J} \gamma^{n'_j} \delta^{t'_j}$ is

such that $x \succeq y$, we obtain

$$\min_{j \in J} n'_j \geqslant \min_{i \in I} n_i$$

and hence $\forall j \in J$, $n'_j \geq 0$. We conclude that y has no negative exponent in γ and consequently y admits a causal approximation. ∎

IV. CONTROLLERS SYNTHESIS FOR CONSTRAINED SYSTEMS

A. Problem statement

Considering DEDS modeled by their state equation (6), we are interested in the synthesis of *state feedback* controllers for $(max, +)$-linear systems. More precisely, if we consider a DEDS modeled by a TEG, using its transfer representation in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$, we will compute a transfer for the feedback controller. This transfer will be realized by a TEG and the application of the controller will lead to merge the TEG of the controller with the TEG of the system. In this controlled TEG, the additional arcs due to the controller authorize or prohibit the firing of the controlled transitions (see figure 1). This control structure is comparable with some Petri nets methods for controlled DEDS [9].

The synthesis of feedback for TEG has previously been tackled in papers such as [5], [12]. In these works, the feedback is aimed at delaying events in the system as much as possible such that the controlled system is not slower than a reference model.

In this paper, the control objective is different :

- we aim at ensuring some given constraints on the state x (rather than satisfying a reference model matching) for all inputs. These constraints are defined by a matrix ϕ and are formulated by the implicit inequality :

$$\phi x \preceq x. \tag{7}$$

- we look for a feedback which delays the functioning of the system as less as possible (that is, which postpones input events as less as possible, opposed to just-in-time criterion). In other words, we aim at computing the least feedback such that the state of the controlled system satisfies the constraints given by (7).

In the following, we illustrate three constraints which can be imposed on the controlled systems as an inequality (7). Next, the control problem is formalized and solved as a state feedback synthesis.

B. Constraints specification

We now detail three kinds of constraints for DEDS described by a TEG, that can be formulated by inequality (7):

- Some inner variables can be subject to a minimum time separation between two successive firings. For a state variable x_i and a time separation denoted Δ_{min}, we require that $x_i(k + 1) \succeq \Delta_{min} \, x_i(k)$. Then, the counterpart of this constraint in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is $\gamma \delta^{\Delta_{min}} x_i \preceq x_i$.

- We can also aim at bounding the sojourn times of tokens in given paths of a TEG (critical time constraints). Let us consider a path from transition x_i to transition x_j containing α tokens initially and we denote τ the desired maximum sojourn time in this path. This yields $x_j(k + \alpha) - x_i(k) \preceq \tau$, which can be formulated in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ by $\gamma^{-\alpha} \delta^{-\tau} x_j \preceq x_i$.

- We may also limit the number of tokens in some paths of a TEG. Let us consider a path from x_i to x_j containing α tokens initially, we denote κ the desired maximum number of tokens in this path. This constraint can be specified by $\gamma^{\kappa-\alpha} x_j \preceq x_i$ in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$.

C. Formalization

We consider a state feedback controller. In this structure a controller, denoted by F, is added between internal state x_c and input u. The process input is described by $u = Fx_c \oplus v$, with v the reference input. Such a controller implies that the delayed events are only the inputs one. The concerned variables are the ones belonging to the set $\mathcal{U}_c = \{u_i | B_{ii} = e\}$. The state evolution of the controlled system is then described by

$$x_c = Ax_c \oplus BFx_c \oplus Bv.$$

By considering the earliest functioning rule, the transfer relation of such controlled system is

$$x_c = (A \oplus BF)^* Bv = H_c v. \tag{8}$$

Remark 2: Assumption 1 implies that the feedback on inputs has an effect on the state variables that are directly controllable, these state variables x_i such that $B_{ii} = e$. These state variables x_i are such that $x_i = u_i$, since there is no shift between them. We denote this set by $\mathcal{X}_c = \{x_i | B_{ii} = e\}$.

From (8), it is obvious that the state of the controlled system is such that $x_c \succeq A^* Bv, \forall v$. Furthermore, x_c should satisfy the control objective (7), *i.e.*, $x_c \succeq \phi x_c$, then $x_c \succeq A^* Bv \oplus \phi x_c, \forall v$.

We aim at delaying the system as less as possible, therefore we seek the least controlled x_c given by $x_c \succeq \phi^* A^* Bv, \forall v$. Using (8), we then look for the least feedback F such that

$$(A \oplus BF)^* Bv \succeq \phi^* A^* Bv, \quad \forall v,$$

$$\Leftrightarrow (A \oplus BF)^* B \succeq \phi^* A^* B. \tag{9}$$

We can easily prove that (9) is equivalent to

$$(A \oplus BF)^* B \succeq \phi^+ A^* B, \tag{10}$$

in which $\phi^+ = \phi\phi^* = \bigoplus_{i \geq 1} \phi^i$ and therefore $\phi^* = e \oplus \phi^+$. We have $(9) \Rightarrow (10)$ since $\phi^* \succeq \phi^+$, and consequently $\phi^* A^* B \succeq \phi^+ A^* B$, and $(10) \Rightarrow (9)$:

$$
\begin{aligned}
& (A \oplus BF)^* B & \succeq \quad & \phi^+ A^* B \\
\Rightarrow \quad & (A \oplus BF)^* B \oplus A^* B & \succeq \quad & \phi^+ A^* B \oplus A^* B \\
\Rightarrow \quad & (A \oplus BF)^* B \oplus A^* B & \succeq \quad & \phi^* A^* B & \text{since } \phi^+ A^* B \oplus A^* B = \phi^* A^* B \\
\Rightarrow \quad & (A \oplus BF)^* B & \succeq \quad & \phi^* A^* B & \text{since } (A \oplus BF)^* B \succeq A^* B.
\end{aligned}
$$

Assumption 2: The matrix of constraints ϕ is supposed to satisfy $B\phi = \phi$.

This assumption comes down to formulating all constraints $\phi_{ij} x_j \preceq x_i$ (see §IV-B) such that $x_i \in \mathcal{X}_c$, that is, on states x_i that are directly controllable. For a given constraint on a path between two state variables $x_i \notin \mathcal{X}_c$ and $x_j \notin \mathcal{X}_c$, our approach requires to recast the constraint such that $x_i \in \mathcal{X}_c$ or $x_j \in \mathcal{X}_c$. For example, consider the TEG of fig.1 and suppose that tokens must not sojourn more that 4 units of time in the path between transitions x_3 and x_5. Hence, it leads to a matrix ϕ that does not satisfy assumption 2. Nonetheless, it is possible to recast this constraint in a way that assumption 2 is satisfied. Instead of considering the path between x_3 and x_5, we can select the paths between x_1 and x_5 and a maximum sojourn time of 5 unit times. This new constraint implies the original one but also implies a new constraint for the sojourn time between x_1 and x_3 (at most 2 units of time). Note that, another possibility is to consider the path between x_2 and x_5 and a maximum sojourn time of 6 units of time.

The following proposition gives a necessary and sufficient condition on the given constraints ϕ for the existence of a causal feedback satisfying (10).

Proposition 8: There exists a causal feedback F satisfying (10) if, and only if, $\phi^+ A^* B$ admits a causal upper approximation. If it exists, the causal upper approximation is denoted by G.

Proof:

$\Rightarrow$ If a causal feedback F exists, then $(A \oplus BF)^* B$ is also causal (since A and B are causal). We can derive from corollary 2 and (10) that $\phi^+ A^* B$ admits a causal upper approximation.

$\Leftarrow$ If $\phi^+ A^* B$ admits a causal upper approximation, then one can find a causal element X such that $X \succeq \phi^+ A^* B$. Since $B^2 = B$ and $B\phi^+ = B(\phi \oplus \phi\phi \oplus ...) = \phi^+$, we have

$$
\begin{aligned}
& BXB && \succeq \phi^+ A^* B && \\
\Rightarrow\quad & (BX)^* B && \succeq \phi^+ A^* B && \text{since } a^* \succeq a \\
\Rightarrow\quad & (BX)^* B \oplus A^* B && \succeq \phi^+ A^* B && \\
\Rightarrow\quad & ((BX)^* \oplus A^*)B && \succeq \phi^+ A^* B && \\
\Rightarrow\quad & ((BX) \oplus A)^* B && \succeq \phi^+ A^* B && \text{since } (a \oplus b)^* \succeq a^* \oplus b^*
\end{aligned}
$$

which proves that a causal feedback (here denoted X) satisfying (10) exists.

■

Corollary 3: The causal upper approximation G, if it exists, is such that $GB = G$ and $BG = G$.

Proof: We first demonstrate that $GB = G$. Since $B \preceq e$, we have $GB \preceq G$. From proposition 8, G is such that $G \succeq \phi^+ A^* B$ and we have $GB \succeq \phi^+ A^* B$ ($B^2 = B$). The matrix GB is causal since G and B are, and as G is the least causal element greater than $\phi^+ A^* B$, we deduce $GB \succeq G$. By the same reasoning, we can easily prove $BG = G$. ■

Remark 3: In [13], considering the dater functions on a prediction horizon, (max,+)-linear systems are described by the state equation (5) and the output equation $y(k) = Cx(k)$. The MPC is extended to this class of systems by defining a control horizon, a cost criterion as well as constraints given by

$$E(k)u(k) + F(k)y(k) \leq h(k), \tag{11}$$

in which $E(k)$, $F(k)$ and $h(k)$ (matrices of adequate dimensions) are chosen according to the control goals. The MPC has been considered for the just-in-time output tracking problem in [13], and it can also be applied to the present control problem. Let us mention the outlines of such a formulation:

- Consider a cost criterion which leads to a minimization of the input time instants.

- Assume that each constraint expressed by (7) applies between an input and an output of the system to recast it (with a possible increase of the prediction horizon) as a constraint in (11).

- Take into account the reference input v as inequality $-u(k) \leq -v(k)$ compatible with (11).

Formulated this way, that is as a nonlinear convex optimization problem, several algorithms have been proposed to solve the MPC problem. It should be clear that this solution depends on the reference input v. This implies that v must be known (at least on the prediction horizon) and that the control law must be implemented online. In contrast, the feedback proposed in this paper is computed offline (using proposition 9) and is valid for all possible reference input v (i.e., v is assumed to be unknown). Furthermore, with the MPC approach, each constraint must apply between an input and an output of the system (see the second item above) and this is more restrictive than assumption 2.

D. Feedback computation

In this section, we investigate how to compute a solution of (10).

Proposition 9: Suppose that $\phi^+ A^* B$ admits a causal upper approximation denoted G (necessary and sufficient condition for the existence of a causal feedback satisfying (10)). Solutions of (10) are elements of $\mathcal{Q}_g$ (see Notation 1) with $g : F \mapsto B(G \ominus (A \oplus BF)^*)$.

Proof: Causal feedbacks used are such that

$$
\begin{aligned}
& (A \oplus BF)^* B && \succeq & G & \\
\Leftrightarrow\ & (A \oplus BF)^* && \succeq & G && \text{(since } GB = G \text{ and } B \preceq e) \\
\Leftrightarrow\ & BF \oplus (A \oplus BF)^* && \succeq & G && \text{(since } (A \oplus BF)^* \succeq BF) \\
\Leftrightarrow\ & BF && \succeq & G \ominus (A \oplus BF)^* && (T_{(A\oplus BF)^*} \text{ is dually residuated}) \\
\Leftrightarrow\ & F && \succeq & B(G \ominus (A \oplus BF)^*). &
\end{aligned}
$$

For the last equivalence :

$$
\begin{aligned}
(\Rightarrow)\quad BF \succeq G \ominus (A \oplus BF)^* \ &\Rightarrow\ B^2 F \succeq B(G \ominus (A \oplus BF)^*) \\
&\Rightarrow\ \ \ F \succeq B(G \ominus (A \oplus BF)^*) \qquad \text{(since } F \succeq BF = B^2 F)
\end{aligned}
$$

$$(\Leftarrow) \quad F \succeq B(G \obardot (A \oplus BF)^*) \quad \Rightarrow \quad BF \succeq B^2(G \obardot (A \oplus BF)^*)$$
$$\Rightarrow \quad BF \succeq B(G \obardot (A \oplus BF)^*)$$
$$\Rightarrow \quad BF \succeq BG \obardot B(A \oplus BF)^* \qquad (\text{since } a(x \obardot b) \succeq ax \obardot ab)$$
$$\Rightarrow \quad BF \succeq BG \obardot (A \oplus BF)^* \qquad (x \mapsto a \obardot x \text{ is antitone})$$
$$\Rightarrow \quad BF \succeq G \obardot (A \oplus BF)^* \qquad (BG = G).$$

$\blacksquare$

Corollary 4: The computation of $\nu = Sup\ \mathcal{F}_{g^2}$ (using Proposition 3) gives a feedback ensuring (10) (since $\nu \in \mathcal{Q}_g$, see prop. 6).

Remark 4: To summarize, Proposition 8 gives a necessary and sufficient condition for the existence of a solution to our control problem. If it is satisfied, then Corollary 4 states how to compute a solution, namely $\nu = Sup\ \mathcal{F}_{g^2}$. As pointed out in remark 1, ν is a good solution to our control problem since it approximates or corresponds to a minimal feedback.

E. Example

We consider the DEDS modeled by the TEG in fig.1 and whose representation is given in section III-A.

To begin with, we will illustrate that not all constraints defined as in IV-B are suitable. Nonetheless, as stated in proposition 8, the computation of $\phi^+ A^* B$ enables us to detect unsuitable constraints if it contains at least one entry with a negative exponent in γ. For example, bounding the sojourn time of tokens in the path between transitions x_2 and x_5 leads to a matrix $\phi^+ A^* B$ containing an entry with a negative exponent in γ. A natural explanation is that if transition u_1 is never fired (the feedback is designed for all possible inputs), then the token initially in the place between x_2 and x_3 will remain indefinitely in the considered path. In this case, any relevant feedback cannot be found.

We now consider suitable constraints:

- tokens must not sojourn more than 5 time units in the paths between transitions x_1 and x_5, then $\delta^{-5} x_5 \preceq x_1$,
- the number of tokens in the path between x_2 and x_4 must not exceed 3, hence $\gamma^2 x_4 \preceq x_2$.

We have $\phi_{15} = \delta^{-5}$, $\phi_{23} = \gamma^2$ and $\phi_{ij} = \varepsilon$ otherwise ($1 \le i, j \le 5$).

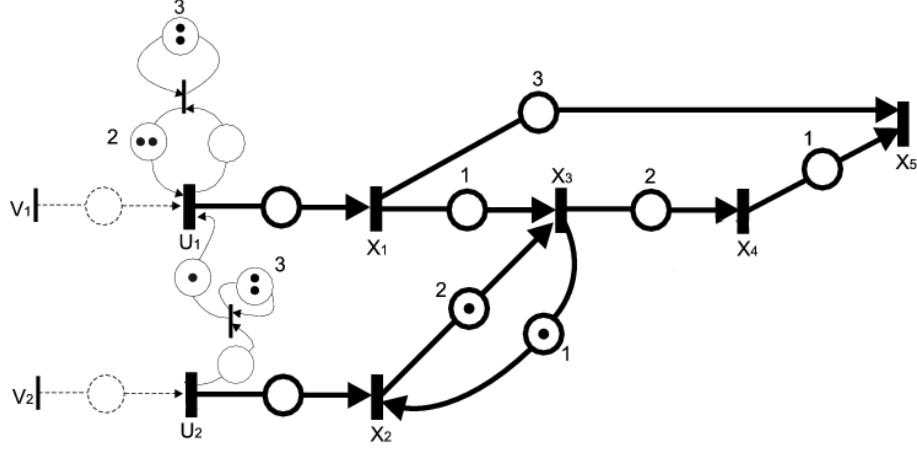According to §IV-D, we can compute the following feedback

Fig. 1.   A TEG (thick lines) merged with a realization of its controller (thin lines) and the (external) reference inputs (dotted lines).

$$F = \nu = \begin{pmatrix} \gamma^2\delta^2(\gamma^2\delta^3)^* & \gamma^1(\gamma^2\delta^3)^* & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

which satisfies both constraints. A realization of this controller is represented in thin lines in fig.1. The computation of F has been implemented with the C++ library `libminmaxgd` [2] handling formal power series in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$.

Let us note that, for this example, we have $\nu \neq \mu$, and hence we cannot argue, thanks to corollary 1, that ν is a minimal feedback. In fact, there exists a feedback F', defined by $F'_{ij} = v_{ij}$ for $(i, j) \neq (1, 1)$ and $F'_{11} = \varepsilon$, which is less than F and which satisfies (10). Nevertheless, let us point out that the controlled system with F' has the same transfer as the controlled system with F, that is $(A \oplus BF')^*B = (A \oplus BF)^*B$ (see Eq. (8)). This means that delays are equally

[2]www.istia.univ-angers.fr/~hardouin/outils.html

minimized by F' and F. This observation reinforces our suggestion that $F = \nu$ constitutes a good approximated solution for our control problem (see remark 4).

V. Conclusion

We have presented a new control problem in $(max, +)$-linear system theory: ensure some given constraints while delaying the systems as less as possible. Using results on antitone and isotone mappings, we propose a state feedback. It must be noted that the controller obtained is not necessarily minimal. In the future, we will focus our attention on improvements of our control approach in that sense.

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