Controllability of (max,+) Formal Power Series ⋆

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Abstract: Controllability of (max,+) automata and formal power series is studied within a behavioral framework. An extension of classical tensor product of their linear representations as a parallel composition of controller with the plant (max,+) automaton is used. Controllability is studied using residuation theory of (multivariable) formal power series and (max,+)-counterpats of supremal controllable behaviors are derived.

Keywords: Controllability, (max,+) automata, (max,+) formal power series, Hadamard product

1. INTRODUCTION

(Max,+) automata model an important class of Timed Discrete Event (dynamical) Systems (TDES), where both synchronization of tasks and resource sharing occur. They have been proposed by S. Gaubert in (4) as weighted automata with weights (multiplicities) in the $(R \cup \{-\infty\}, \max, +)$ semiring. (max,+) automata have a strong expressive power in terms of timed Petri nets: 1-safe timed Petri net can be represented by special (max,+) automata, called heap models, c.f. (5).

We have proposed a behavioral approach (based on formal power series) to supervisory control of (max+)-automata in (8). It is based on the parallel composition of controller and plant (max,+)-automata with uncontrollable events. This composition corresponds to a modified (due to uncontrollable events) version of tensor product in terms of linear representation in the $(R \cup \{-\infty\}, \max, +)$ semiring and to a generalized Hadamard product (distinguishing uncontrollable events) in terms of behaviors.

In this paper, we build upon these results and investigate the properties of this generalized Hadamard product of a controller and plant formal power series. Controllability as an equivalent condition for attainability of a specification series as the prescribed behavior of the closed-loop system is studied using residuation theory of (multivariable) formal power series. A formula for computing (max,+)-counterparts of supremal controllable behavior is proposed. An interesting comparison with controllability of formal languages (from classical supervisory control theory) is given together with intuition behind timing aspects of controllability.

This paper is organized as follows. In the next section necessary algebraic preliminaries are recalled together with parallel composition of (max,+) (weighted) automata. In Section 3 we recall the behavioral framework, where parallel composition of (max,+) automata corresponds to a generalized Hadamard product of formal power series and a control problem is posed that takes into account both nondecreasing and general formal power series (counterpart of prefix closed and marked languages). In section 4 controllability and properties of controllable formal power series are studied using residuation theory. An illustrating example is proposed. Conclusion with hints on future extensions of this work are given in Section 5.

2. (MAX,+)-AUTOMATA AND THEIR PROPERTIES

In this section necessary algebraic concepts are recalled. An *idempotent semiring* (also called dioid) is a set $\mathcal{D}$ equipped with two binary operations: addition and multiplication. The addition $\oplus$ is commutative, associative, has a unit element ε (i.e. $\varepsilon \oplus a = a$ for each $a \in M$), and is idempotent (i.e. $a \oplus a = a$ for each $a \in M$). The multiplication $\otimes$ is associative, has a unit element e, and distributes over $\oplus$. Moreover, ε is absorbing for $\otimes$, i.e. $\forall a \in M: \ a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$.

In any dioid, a natural order is defined by: $a \preceq b \Leftrightarrow a \oplus b = b$. A dioid $\mathcal{D}$ is complete if each subset A of $\mathcal{D}$ admits a least upper bound denoted $\bigoplus_{x \in A} x$, and if $\otimes$ distributes with respect to infinite sums. In particular, $T = \bigoplus_{x \in \mathcal{D}} x$ is the greatest element of $\mathcal{D}$. In a complete dioid, the greatest lower bound, denoted by $\wedge$, always exists; $a \wedge b = \bigoplus_{x \preceq a, x \preceq b} x$.

Let us recall the dioid $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ with maximum playing the role of addition, denoted by $\oplus$: $a \oplus b = \max(a, b)$, and conventional addition playing the role of multiplication, denoted by $a \otimes b$ (or ab when unambiguous). Its complete version with $T = +\infty$ added is denoted by $\overline{\mathbb{R}}_{\max}$. Operations with matrices are defined as in the classical linear algebra. The (max,+) identity matrix of $\mathbb{R}_{\max}^{n \times n}$ is denoted by E. Let $\mathbb{N}$ denote the set of natural numbers with zero. In complete dioids the star operation can be introduced by the formula

$$a^* = \bigoplus_{n \in \mathbb{N}} a^n,$$

where by convention $a^0 = e$ for any a.

Residuation theory allows defining 'pseudo-inverses' of isotone maps (f is isotone if $a \preceq b \Rightarrow f(a) \preceq f(b)$).

Definition 2.1. (2), §4.4.4 An isotone map $f : \mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ and $\mathcal{C}$ are dioids, is said to be residuated if there exists an isotone mapping $h : \mathcal{C} \to \mathcal{D}$ such that

$$f \circ h \preceq Id_{\mathcal{C}} \text{ and } h \circ f \succeq Id_{\mathcal{D}}. \quad (1)$$

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$Id_{\mathcal{C}}$ and $Id_{\mathcal{D}}$ are identity maps of $\mathcal{C}$ and $\mathcal{D}$ respectively. h is unique, it is denoted $f^\sharp$ and is called residual of f.

If f is residuated then $\forall y \in \mathcal{C}$, the least upper bound of subset $\{x \in \mathcal{D} | f(x) \preceq y\}$ exists and belongs to this subset. It is equal to $f^\sharp(y)$. We recall from (2) the following result.

Theorem 2.1. $f : C \rightarrow D$ between two complete dioids is residuated iff

(i) $f(\varepsilon) = \varepsilon$, and
(ii) f is lower semicontinuous, i.e. $f(\oplus_{i\in I} x_i) = \oplus_{i\in I} f(x_i)$.

It is well known that multiplication in complete dioids is residuated.

Theorem 2.2. The isotone map $R_a : x \mapsto x \otimes a$ in a complete dioid $\mathcal{D}$ is residuated. The greatest solution of $x \otimes a \preceq b$ exists and is equal to $R_a{}^\sharp(b)$, also denoted $b \not\!\!\! \circ a$.
This 'quotient' satisfies the following formulae

$$(x \not\!\!\! \circ a) \otimes a \preceq x, \qquad (f.1)$$
$$(x \otimes a) \not\!\!\! \circ a \succeq x. \qquad (f.2)$$

Formal languages over an alphabet A are sets of finite sequences of letters (called words) from A. The zero language is $0 = \{\}$, the unit language is $1 = \{\varepsilon\}$ with ε the empty string. A string $u = u_1 \ldots u_k \in A^*$ is called a subword of $v \in A^*$ if there exists a factorization $v = v_1 u_1 v_2 \ldots v_k u_k v_{k+1}$ with $v_i \in A^*,\ i = 1, \ldots k+1$. The induced subword order on A^* is $u \preceq v$ iff u is a subword of $v \in A^*$.
The dioid of formal power series with variables from A and coefficients from $\mathbb{R}_{\max}$, endowed with point-wise addition and convolution multiplication, is denoted by $\mathbb{R}_{\max}(A)$. Thus, for $s = \oplus_{w\in A^*} s(w) w \in \mathbb{R}_{\max}(A)$ and $s' = \oplus_{w\in A^*} s'(w) w \in \mathbb{R}_{\max}(A)$, one has:

$$s \oplus s' \triangleq \oplus_{w\in A^*} (s(w) \oplus s'(w)) w \ ,$$
$$s \otimes s' \triangleq \oplus_{w\in A^*} (\oplus_{uv=w} s(u) \otimes s'(v)) w.$$

$\mathbb{R}_{\max}(A)$ is isomorphic to the dioid of generalized dater functions from A^* to $\mathbb{R}_{\max}$. The dioid of formal power series is complete if we work with coefficients in $\overline{\mathbb{R}}_{\max}$. Notice that for $s, s' \in \mathbb{R}_{\max}(A)$, $s \preceq s'$ (natural order on $\mathbb{R}_{\max}(A)$) amounts to $s(w) \preceq s'(w)$ for all $w \in A^*$. The language $supp(s) = \{w \in A^* :\ s(w) \neq -\infty\}$ is called the support of the series s. Recall that a formal power series is recognized by a finite (max,+) automaton iff it is rational, i.e. iff it can be formed by rational operations from polynomial series (those with finite support).

Another multiplication of series (element-wise or word by word), called Hadamard product, will be needed and is defined by:

$$s, s' \in \mathbb{R}_{\max}(A),\ s \odot s' \triangleq \oplus_{w\in A^*} (s(w) \otimes s'(w)) w.$$

It has been shown in (8) that Hadamard product, denoted $H_y : \overline{\mathbb{R}}_{\max}(A) \rightarrow \overline{\mathbb{R}}_{\max}(A), s \mapsto s \odot y$ is residuated.

Proposition 2.3. The mapping H_y: $\overline{\mathbb{R}}_{\max}(A) \rightarrow \overline{\mathbb{R}}_{\max}(A)$, $s \mapsto s \odot y$ is isotone, residuated, and its residual is given by

$$H_y^\sharp(s)(w) = s(w) \not\!\!\! \circ y(w), \qquad (2)$$

i.e. $H_y^\sharp(s) = \bigoplus_{w\in A^*} (s(w) \not\!\!\! \circ y(w)) w$.

Even more, Hadamard product admits an inverse, which is known as Hadamard quotient in the theory of formal power series over rings. However, a generalized version of Hadamard product, defined and used further in this paper, is only residuated. Hence, the notation of residuation theory is kept also for H_y.

Natural projections of languages are now recalled and extended to formal power series. The natural projection from A^* to A_c, where $A_c \subseteq A$ is denoted by P_c.

It projects away from any string $w \in A^*$ events from $A_u = A \setminus A_c$, *cf.* (10). Formally, $P_c : A^* \rightarrow A_c^*$ is defined as follows on events from A

$$P_c(a) = \begin{cases} a & \text{if } a \in A_c \\ \varepsilon & \text{if } a \in A \setminus A_c \end{cases}$$

and P_c is extended to words in such a way that P_c is catenative: $P_c(a_1 \ldots a_n) = P_c(a_1) \ldots P_c(a_n)$. Similarly, P_c is extended to languages (subsets of A^*) in an obvious way: for $L \subseteq A^*$: $P_c(L) = \cup_{w\in L} P_c(w) \subseteq A_c^*$. In the sequel A_c and A_u play the role of controllable and uncontrollable events, respectively.

A notion of projection of formal power series will be needed.

Definition 2.2. For any formal power series $s = \oplus_{w\in A^*} s(w) w \in \mathbb{R}_{\max}(A)$ and $A_c \subseteq A$ with the associated natural projection $P_c :\ A^* \mapsto A_c^*$ we associate the projected series $P(s)$ given by the following coefficients:

$$P(s)(w) = s(P_c w).$$

Let us note the difference between $P(s)$ and the following formal power series: $\tilde{P}(s) = \oplus_{w\in A^*} s(w) P_c w \in \mathbb{R}_{\max}(A)$. It is easily seen on the series supports (that are languages). While the operator $\tilde{P}(s)$ can only decrease the support, our operator $P(s)$ can only increase the support. In particular, let us notice that P has values in $\mathbb{R}_{\max}(A)$ and not in $\mathbb{R}_{\max}(A_c)$. For instance, if $A_c = \{a\} \subseteq \{a, u\} = A$ and $s = 1 \oplus 2a$ then $P(s) = 1u^* \oplus 2u^* a u^*$. Indeed, we have by definition $P(s)(\varepsilon) = P(s)(u) = P(s)(u^2) = \cdots = s(\varepsilon) = 1$ and similarly, $P(s)(w) = s(a) = 2$ for any $w \in u^* a u^*$. Hence, our operator $P :\ \mathbb{R}_{\max}(A) \rightarrow \mathbb{R}_{\max}(A)$ is not compatible with projection on words and languages (it is not the morphic extension of P_c).

On the other hand, natural projection $P_c : A^* \rightarrow A_c^*$ can be extended to $\tilde{P} :\ \mathbb{R}_{\max}(A) \rightarrow \mathbb{R}_{\max}(A_c)$ by the formula $\tilde{P}(s) = \oplus_{w\in A^*} s(w) P_c w \in \mathbb{R}_{\max}(A)$, which corresponds to coefficients given by $\tilde{P}(s)(P_c w) = s(w)$, i.e. $\tilde{P}(s)(w) = \oplus_{u\in P_c^{-1}(w)} s(u)$. This is to explain the difference between $\tilde{P}$ and $P : \mathbb{R}_{\max}(A) \rightarrow \mathbb{R}_{\max}(A)$ of Definition 2.2.

Finally, basic definitions of tensor products are recalled.

If $A = (a_{ij})$ is a $m \times n$ matrix and B is a $p \times q$ matrix over a dioid, then their *Kronecker (tensor) product* $A \otimes^t B$ is the $mp \times nq$ block matrix

$$A \otimes^t B = \begin{bmatrix} a_{11} \otimes B & \cdots & a_{1n} \otimes B \\ \vdots & \ddots & \vdots \\ a_{m1} \otimes B & \cdots & a_{mn} \otimes B \end{bmatrix}$$

Now we recall automata with multiplicities in the $\mathbb{R}_{\max}$ semiring, called (max,+) automata (4).

Definition 2.3. A (max,+) automaton over an alphabet A is a quadruple $G = (Q, \alpha, t, \beta)$, where Q is a finite set of states, $\alpha : Q \rightarrow \mathbb{R}_{\max}$, $t : Q \times A \times Q \rightarrow \mathbb{R}_{\max}$, and $\beta : Q \rightarrow \mathbb{R}_{\max}$, called input, transition, and output delays, respectively.

The transition function associates to a state $q \in Q$, a discrete input $a \in A$ and a new state $q' \in Q$, an output value $t(q,a,q') \in \mathbb{R}$ corresponding to the $a-$transition from q to q' or $t(q,a,q') = \varepsilon$ if there is no transition from q to q' labeled by a. The real output value of a transition is interpreted as the duration of this transition.

A (max,+) automaton is determined by a triple (α,μ,β), where $\alpha \in \mathbb{R}_{\max}^{1\times Q}$, $\beta \in \mathbb{R}_{\max}^{Q\times 1}$ and μ is a morphism defined by:

$$\mu : A \to \mathbb{R}_{\max}^{Q\times Q},\ \mu(a)_{q\,q'} \triangleq t(q,a,q').$$

We will call such a triple a linear representation.

Note that the morphism matrix μ of a (max,+) automaton can also be considered as an element of $\mathbb{R}_{\max}(A)^{Q\times Q}$, *i.e.* $\mu = \oplus_{w\in A^*}\mu(w)w$ by extending the definition of μ from $a \in A$ to $w \in A^*$ using the morphism property

$$\mu(a_1 \dots a_n) = \mu(a_1)\dots\mu(a_n).$$

Recall that μ has an important property of being finitely generated, because it is completely determined by its values on A. Hence we have in fact $\mu^* = (\oplus_{a\in A}\mu(a)a)^*$. Since we are interested in behaviors of (max,+) automata that are given by $l = \alpha\mu^*\beta$ (see below) we abuse the notation and simply write $\mu = \oplus_{a\in A}\mu(a)a$.

Since the plan is to extend the supervisory control techniques from logical to (max,+) automata, it is useful to formulate (max,+) automata in standard automata description (using initial and final states).

For purposes of supervisory control it is useful to see a (nondeterministic) (max,+) automaton over an event alphabet A as the 4-tuple $G = (Q, q_0, Q_m, t)$, where Q is the set of states, q_0 is the initial state, Q_m is the subset of final or marked states, and $t : Q \times A \times Q \to \mathbb{R}_{\max}$ is the (possibly nondeterministic) transition function with inputs in A and outputs in $\mathbb{R}_{\max}$.

However, the last definition does not consider nonzero initial delays, resp. final delays : these are only Boolean and equal to e *iff* the corresponding state is initial, resp. final.

The formal power series recognized by a (max,+) automaton $G = (Q,\alpha,t,\beta)$, called its behaviour, is given by $l(G) : A^* \to \mathbb{R}_{\max}$ defined for $w = a_1 \dots a_n \in A^*$ by

$$l(G)(w) = \max_{q_0,\dots,q_n\in Q} \alpha(q_0) \otimes \left[\sum_{i=1}^{n} t(q_{i-1},a_i,q_i)\right] \otimes \beta(q_n). \tag{3}$$

In words, $l(G)(w)$ is the maximal weights of paths labeled by w going from the initial state to a final state.

Remark 2.4. The series $l(G) : A^* \to \mathbb{R}_{\max}$ is a dater (4). We shall interpret $l(G)(w)$ as the time of completion of the sequence of events w, with the convention that $l(G)(w) = -\infty = \varepsilon$ if w does not occur. By specialization to "boolean" series with values in $\{\varepsilon, e\}$, we obtain the classical interpretation of Ramadge and Wonham theory, that is $l(G)(w) \neq \varepsilon$ if w corresponds to an admissible behavior of the system.
By extension, to study logical aspects of $(\max,+)$ automata it is sufficient to work with supports of series corresponding to behaviors. We shall then consider series with boolean coefficients (in $\{\varepsilon, e\}$) instead of $\mathbb{R}_{\max}$ (any coefficient different from ε becomes e).

In terms of linear representation : $l(G)(w) = \alpha \otimes \mu(w) \otimes \beta$.

Similarly as timed event graphs are described by fixed point equations in the dioid of formal power series $Z_{\max}(\gamma)$ of (2, §5.3), any (max,+) automaton is described by the following fixed point equation in the dioid $\mathbb{R}_{\max}(A)$ of formal power series with non commutative variables from A:

$$x = x\mu \oplus \alpha \tag{4}$$

$$y = x\beta, \tag{5}$$

with $\mu = \bigoplus_{a\in A}\mu(a)a \in \mathbb{R}_{\max}(A)^{|Q|\times|Q|}$ the morphism matrix.

It is known that the least solution to this equation is $y = \alpha\mu^*\beta$.

The parallel composition below is defined as an extension of parallel composition (synchronous product) from logical to timed DES. The first automaton plays the role of the controller and the second is the system (to be controlled). As usual in supervisory control, $A = A_c \cup A_u$ is the partition of event set A into disjoint subsets of controllable and uncontrollable events, respectively.

Definition 2.4. Consider the two following (max,+) automata corresponding to the controller and the system:

$$G_c = (Q_c, q_{c,0}, Q_m^c, t_c),\ G = (Q_g, q_{g,0}, Q_m^g, t_g). \tag{6}$$

Their *parallel composition*, modeling the system under control, is

$$\begin{gathered} G_c\|_{A_u} G = (Q_c \times Q_g, q_0, Q_m, t) \\ \text{with } q_0 = \langle q_{c,0}, q_{g,0}\rangle,\ \ Q_m = Q_c \times Q_m^g, \\ t(\langle q_c, q_g\rangle, a, \langle q_c', q_g'\rangle) = \\ \begin{cases} t_c(q_c,a,q_c') \otimes t_g(q_g,a,q_g'), & \text{if } a \in A_c \\ t_g(q_g,a,q_g'), & \text{if } a \in A_u \text{ and } q_c = q_c' \\ \varepsilon, & \text{if } a \in A_u \text{ and } q_c \neq q_c' \end{cases} \end{gathered} \tag{7}$$

This definition can be seen as an extension of prioritized synchronous composition of (6) or (7) from Boolean to the (max,+) case. Let us stress that it expresses the intuitive requirement that the controller automaton can not disable an uncontrollable event that occurs in the plant. Similarly as in the classical supervisory control the controller can not unmark the marked states of the original system : for any state that is marked in the original plant G and survives the logical supervision, the corresponding state in $G_c\|_{A_u}G$ is marked. This means that marked states of the controller do not play any role and may be ignored, which is expressed by $Q_m = Q_c \times Q_m^g$. In the sequel we can then assume that all states of the controller are marked without loss of generality.

Controllable transitions (i.e. $t_g(q_g,a,q_g'),\ \ a \in A_c$) in the plant G can be in the composed system $G_c\|_{A_u}G$ both disabled (due to ε absorbing for multiplication : when the synchronizing transition of the controller is not defined $t_c(q_c,a,q_c') = \varepsilon$) and delayed (when $t_c(q_c,a,q_c') > 0$). The delay is added to the duration of the corresponding transition in $G_c\|_{A_u}G$. On the other hand, uncontrollable transitions (i.e. $t_g(q_g,a,q_g'),\ \ a \in A_u$) in the plant G can be in the composed system $G_c\|_{A_u}G$ neither disabled nor delayed.

The interpretation of the parallel composition of a system with its controller is as follows. The controller is another (max,+)-automaton running in parallel (in a standard synchronous manner) with the system's automaton, that observes the generated events and either generates the same event as the controller, in which case it may delay the execution of the corresponding transition by the number of time units given by the weights of the transition in the controller (in case of a controllable event) or does not generate this event. In the latter case the event that was possible in the uncontrolled system is disabled in the parallel composition (this event should be controllable in accordance with definition). Uncontrollable events can neither be prevented from happening and can nor be delayed, the uncontrollable transition in the parallel composition inherits the duration from the original uncontrolled plant G.

3. BEHAVIORAL APPROACH TO CONTROL OF (MAX,+) AUTOMATA

In this section the behavioral approach of (9) is recalled and extended.

Proposition 3.1. (9) The *parallel composition* of two (max,+) automata

$$G_c = (\alpha_c, \mu_c, \beta_c),\ G = (\alpha_g, \mu_g, \beta_g). \tag{8}$$

has the following linear representation

$$\begin{aligned} G_c\|_{A_u}G &= (\alpha, t, \beta)\\ \alpha &= \alpha_c \otimes^t \alpha_g,\\ \forall a \in A_c:\quad \mu(a) &= \mu_c(a) \otimes^t \mu_g(a),\\ \forall a \in A_u:\quad \mu(a) &= E \otimes^t \mu_g(a),\\ \beta &= e_c \otimes^t \beta_g, \end{aligned}$$

where $e_c = \beta_c$ denotes the column vector of identity elements $e = 0$ of length given by $|Q_c|$.

Proposition 3.1 is useful for computing the behavior of the composed system consisting of a controller and a plant. It may simply be viewed as an equivalent definition of parallel composition for (max,+) automata in terms of their linear representations that admit nonzero initial and final delays from $\mathbb{R}_{\max}$.

Let us recall from (9) the following theorem about behavior of closed-loop systems.

Theorem 3.2. The behavior of the parallel composition is the following:

$$l(G_c\|G)(w) = l_c(P_c(w)) \otimes l_g(w).$$

By comparing the definition of Hadamard product with the formula of the last theorem we can view the right hand side as a kind of generalized Hadamard product (in presence of uncontrollable events). This is proposed in the following definition.

Definition 3.1. Let $A = A_c \cup A_u$ with the associated natural projection $P_c : A^* \to A_c^*$. The generalized Hadamard product of two formal power series s and s', denoted $\odot_{A_u}$, is defined by $(s \odot_{A_u} s')(w) = s(P_c(w)) \otimes s'(w)$.

It follows from Theorem 3.2 that

$$l(G_c\|G) = l(G_c) \odot_{A_u} l(G) = l_c \odot_{A_u} l_g.$$

This can be applied to control of (max,+) automata in a behavioral framework.

Let y_{ref} be a specification series, the supervisory control problem is to find the greatest controller series, denoted y_C, such that $y_C \odot_{A_u} y \preceq y_{ref}$. Let us introduce the notation

$$H_y^{A_u} : s \mapsto s \odot_{A_u} y$$

for the right generalized Hadamard product.

Let us notice that the mapping $H_y^{A_u}$ is isotone. Since $(H_y^{A_u})^{\uparrow} : \overline{\mathbb{R}}_{\max}^{\uparrow\, supp}(A) \to \overline{\mathbb{R}}_{\max}^{\uparrow\, supp}(A)$ is again a residuated mapping (with its residuated mapping denoted by $(H_y^{A_u})^{\uparrow\,\sharp}$), there exists the greatest y_C such that $(H_y^{A_u})^{\uparrow} \preceq y_{ref}$, namely $y_C^{opt} := (H_y^{A_u})^{\uparrow\,\sharp}(y_{ref})$.

The following Proposition has been proven in (9).

Proposition 3.3. The mapping $H_y^{A_u} : \overline{\mathbb{R}}_{\max}(A) \to \overline{\mathbb{R}}_{\max}(A)$ is residuated and its residuated mapping is given by

$$(H_y^{A_u})^{\sharp}(s)(w) = \tag{9}$$

$$\begin{cases} \bigwedge\limits_{u \in P_c^{-1}(w) \cap supp(y)} ((s(u) \not{\phi} y(u)), & \text{if } w \in A_c^*\\ T, & \text{if } w \notin A_c^* \end{cases}$$

Let us recall that one need not worry about the value $T = \infty$, because when one computes the resulting series, one is interested only in values of projected words, i.e. delays of controllable transitions.

In the next section Proposition 9 will be used in the study of controllability of (max,+) formal power series.

4. CONTROLLABILITY OF (MAX,+) FORMAL POWER SERIES

In the last section the control problem and its solution based on residuation theory have been formulated within a behavioral framework. The resulting series corresponding to an optimal supervisor can then be realized by a (max,+) automaton, provided it is rational.

Similarly as in the classical supervisory control theory not every specification series can be achieved. Since it is not clear how to define controllable (max,+) formal power series, it is natural to define a series to be controllable if it can be exactly achieved by control actions of a suitable supervisor. More precisely, within our behavioral framework we introduce the following concept of controllability.

Definition 4.1. A series $y_{ref} \in \overline{\mathbb{R}}_{\max}(A)$ is controllable with respect to y and A_u if there exists $y_c \in \overline{\mathbb{R}}_{\max}(A)$ such that $y_c \odot_{A_u} y = y_{ref}$, i.e. if $H_y^{A_u}(y_c) = y_{ref}$.

The following characterization of controllability that does not refer to the existence of a controller series, but is based purely on the plant and specification series.

Theorem 4.1. A series $y_{ref} \in \overline{\mathbb{R}}_{\max}(A)$ is controllable with respect to y and A_u iff

$$y_{ref} = H_y^{A_u} \circ (H_y^{A_u})^{\sharp}(y_{ref}).$$

Using the following modified definition of projected formal power series $P_y : \overline{\mathbb{R}}_{\max}(A) \to \overline{\mathbb{R}}_{\max}(A)$ with

$$P_y(s)(w) = \begin{cases} s(P_c(w)), & \text{if } w \in supp(y)\\ \varepsilon, & \text{if } w \notin supp(y) \end{cases}$$

we have in fact $H_y^{A_u} = H_y \circ P_y$, i.e. $\forall s \in \overline{\mathbb{R}}_{\max}(A)$: $H_y^{A_u}(s) = H_y(P_y(s))$. This is because $\otimes$ is absorbing for ε and hence for $w \notin supp(y)$ we can put $P_y(s)(w) = \varepsilon$ without modifying the Hadamard product $H_y^{A_u}(s)(w)$.

The following claim from (9) will be useful.

Proposition 4.2. P_y defined on complete dioids of formal power series is residuated with its residuated mapping given by

$$P_y^\sharp(s)(w) = \begin{cases} \bigwedge_{u \in P_c^{-1}(w) \cap supp(y)} s(u), & \text{if } w \in A_c^* \\ T, & \text{if } w \notin A_c^* \end{cases}$$

Theorem 4.1 provides a useful characterization of controllable series as those that are fixpoints of $H_y^{A_u} \circ (H_y^{A_u})^\sharp$. Note that inequality $H_y^{A_u} \circ (H_y^{A_u})^\sharp(s) \preceq s$ is always satisfied as follows from the very definition of a residuated mapping, c.f. Definition 2.1. Since we have decomposition $H_y^{A_u} = H_y \circ P_y$ using standard Hadamard product H_y (corresponding to the absence of uncontrollable events, i.e. $A_c = A$), we obtain y_{ref} is controllable with respect to y and A_u if and only if

$$y_{ref} = H_y \circ P_y \circ P_y^\sharp \circ H_y^\sharp(y_{ref}).$$

Theorem 4.3. A series $y_{ref} \in \overline{\mathbb{R}}_{\max}(A)$ is controllable with respect to y and A_u iff $\forall w \in A^*$:

$$y_{ref}(w) \not\phi y(w) = \bigwedge_{u \in P_c^{-1} P_c(w) \cap supp(y)} y_{ref}(u) \not\phi y(u).$$

Theorem 4.3 can be reformulated as follows: $\forall w \in A^*$: and $\forall u \in P_c^{-1} P_c(w) \cap supp(y) : \; y_{ref}(u) \not\phi y(u) \succeq y_{ref}(w) \not\phi y(w)$ Otherwise stated, we must have equality for any $w \in supp(y)$, because clearly any such $w \in \{u \in P_c^{-1} P_c(w) \cap supp(y)\}$. Indeed, if $w \notin supp(y)$ then we obtain T on the right (because $a \not\phi \varepsilon$ for any $a \in \overline{\mathbb{R}}_{\max}$, included $a = \varepsilon$) and since $u \in supp(y)$ we obtain T on the left as well. Formally, the following corollary holds true.

Corollary 4.4. A series $y_{ref} \in \overline{\mathbb{R}}_{\max}(A)$ is controllable with respect to y and A_u iff $\forall w \in supp(y)$:

$$\forall u \in P_c^{-1} P_c(w) \cap supp(y) \; : \; y_{ref}(u) \not\phi y(u) = y_{ref}(w) \not\phi y(w).$$

Note that in the characterization of controllability of Corollary 4.4 both logical and timing aspects of controllability are included at the same time. In the sequel both aspects of this characterization will be discussed in details. First of all, timing aspect of controllability is easy to understand. Since y_{ref} as well as y are scalar series, i.e. all coefficients are numbers (including ε), one can reformulate controllability as

$$\forall w \in supp(y) \text{ and } \forall u \in P_c^{-1} P_c(w) \cap supp(y) \; :$$
$$y_{ref}(w) \not\phi y_{ref}(u) = y(w) \not\phi y(u).$$

Note that $u \in P_c^{-1} P_c(w)$ just means that u and w differ only by uncontrollable events. Now, if $w \succeq u$, then the formula expresses the requirement that given a time delay between the occurrence of strings u and w within the system ($y(w) \not\phi y(u)$), the same delay between the strings u and w must be prescribed by the specification series ($y_{ref}(w) \not\phi y_{ref}(u)$). This is a very natural and intuitive requirement, because the intermediate uncontrollable events (that make the difference between those strings : $P_c(u) = P_c(w)$) can not be delayed by any controller automaton.

Let us now define a projection $\tilde{P}_c : A^* \to A^*$ that removes uncontrollable strings (if any) at the end of words. Thus, $\tilde{P}_c(w) = v$ if $w = vu, \; u \in A_u^*$ and $last(v) \in A_c$, where $last(v)$ denotes the last letter of the word v. Then we have

Proposition 4.5. A prefix closed language K is controllable with respect to L and A_u iff $\tilde{P}_c^{-1} \tilde{P}_c(K) \cap L \subseteq K$.

Now we return to the characterization of controllability of series and we extract logical aspects of it to compare with controllability of languages. In this respect, as mentioned in Remark 2.4 it is sufficient to consider the support of series (*i.e.,* series with Boolean coefficients) instead of series having coefficients in $\overline{\mathbb{R}}_{\max}$ (any coefficient different from ε, including T becomes the unit element e). The series y_{ref} plays the role of specification language K, *i.e.,* $y_{ref}(w) = e$ means that $w \in K$ and similarly $y(w) = e$ means that $w \in L$. One can notably check that Proposition 4.5 implies characterization of controllability stated in Corollary 4.4. To do this, let us consider a controllable prefix closed language K and $w \in K$ (*i.e.,* $y_{ref}(w) = e$), from Proposition 4.5 we have $\tilde{P}_c^{-1} \tilde{P}_c(K) \cap L \subseteq K$, and in particular, $\forall u \in \tilde{P}_c^{-1} \tilde{P}_c(w) \cap L$, *i.e.,* $y(w) = e$ and $u \in \tilde{P}_c^{-1} \tilde{P}_c(w)$ which implies $u \in P_c^{-1} P_c(w)$, we have $u \in K$, *i.e.,* $y_{ref}(u) = e$. Then $\forall w \in supp(y)$, *i.e.,* $y(w) = e$, we have the condition of Corollary 4.4, that is

$$y_{ref}(u) \not\phi y(u) = y_{ref}(w) \not\phi y(w) = e \not\phi e = e.$$

The converse implication is not true. More precisely, in the converse reasoning, one can not argue that $u \in P_c^{-1} P_c(w)$ implies $u \in \tilde{P}_c^{-1} \tilde{P}_c(w)$.
This makes a connection between the (max,+) and logical controllability. More precisely, this means that our original notion of controllability for formal power series (with P_c instead of $\tilde{P}_c$) is stronger in its logical aspect than classical R-W controllability of languages. Since there is no notion of prefix closed behaviors for formal power series, the control problem that has been formulated for formal power series that are counterparts of marked languages is more restrictive (*c.f.* for languages inclusion of marked languages implies inclusion of prefix closed languages if the systems are nonblocking). Hence, controllability needs to be stronger.

4.1 Supremal controllable behaviors

If a specification series is not controllable, a natural question is to find an approximation, in particular a smaller series, that is controllable.
Let us first notice that $H_y^{A_u}$ and $(H_y^{A_u})^\sharp$ are isotone mappings. The following result holds.

Proposition 4.6. $H_y^{A_u} \circ (H_y^{A_u})^\sharp(y_{ref})$ is the greatest controllable (max,+) series with respect to y and A_u smaller or equal to y_{ref}.

Remark 4.7. There is an analogy with the classical supervisory control theory. If we denote in the classical supervisory control theory the operator $H_L(K) = \inf \mathrm{C}(K, L, A_u)$ the resulting closed-loop system, which corresponds to the infimal controllable superlanguage of the specification language K with respect to plant language L and A_u, then it can be shown that this mapping is residuated in the dioid of formal languages and its residuated mapping is nothing else but $H_L^\sharp(K) = \sup \mathrm{C}(K, L, A_u)$.

The residuated mapping $(H_y^{A_u}) \circ (H_y^{A_u})^\sharp(s)$ plays the role (i.e. is a generalization of) the supremal controllable sublanguage of specification (reference) series s with respect to the plant y and A_u. Firstly, $H_y^{A_u}(s)$ plays the role of closed-loop behavior of the controlled system. In classical supervisory control it corresponds to the infimal controllable superlanguage. However, in our case, where timing aspect of control is defined by adding delay, i.e. (max,+) multiplication, we can not expect that the supremal controllable subseries of a controllable series is this series itself. Therefore it is not $(H_y^{A_u})^\sharp(s)$, but $(H_y^{A_u}) \circ (H_y^{A_u})^\sharp(s)$ that is the formal power series counterpart of the supremal controllable sublanguage of s. The last proposition can then be viewed as a generalization of the formula for $\sup \mathrm{C}$ operator from Ramadge-Wonham theory.

Another consequence of our investigations of controllability is plausible.

Corollary 4.8. If y_{ref} is controllable with respect to y and A_u then the controller series is simply given by

$$y_c(w) = y_{ref}(w) \not{\circ}\, y(w).$$

Otherwise stated, $(H_y^{A_u})^\sharp(y_{ref}) = H_y^\sharp(y_{ref})$, i.e. the residuation of $H_y^{A_u}$ (for a controllable argument y_{ref}) is simply reduced to the Hadamard quotient.

This is similar to classical supervisory control, where controller is given by the intersection of the plant and the specification languages if the specification is controllable.

The notion of controllability is illustrated in the example below.

Example 1. A manufacturing system modeled by a (max,+) automaton G displayed on figure 1.(a) is considered. The three distinct tasks, labeled a, b and c, last respectively 3, 4 and 5 units of time. The system can perform the following sequences of tasks : a, ab, abc, $abcb$, $abcbc$, ... The behavior of G is given by the following series in $\overline{\mathbb{R}}_{\max}(A)$:

$$y = 3a(9bc)^*(4b + e).$$

For instance, $y(ab) = 7$ means that the sequence ab will be completed at the date 7 (considering that the system starts to operate at time 0).
It is assumed that the start of tasks a and c can be delayed (we may decide to postpone the execution of these tasks when they should be performed) or even forbidden (their execution can be prevented). On the contrary, the task b can neither be delayed nor forbidden (this task starts as soon as it can be performed). Denoting $A = \{a, b, c\}$ the set of events (alphabet), we then have $A_c = \{a, c\}$ and $A_u = \{b\}$.
We would like that the system behaves at the latest according to the following series:

$$y_{ref} = 4a \oplus 9ab \oplus 14abc.$$

This means that the sequences a, ab and abc should be completed at the latest at dates 4, 9 and 14 respectively. In addition, any other sequence of tasks should not happen. This series is recognized by the (max,+) automation G_{ref} displayed on figure 1.(b).

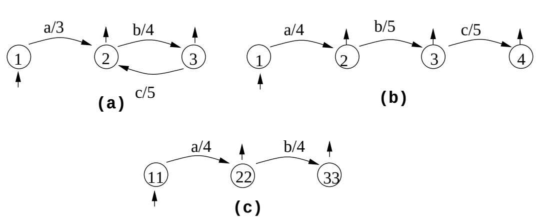

Fig. 1. G **(a)**, G_{ref} **(b)**, G_s **(c)**

The specification y_{ref} is however not controllable with respect to y and A_u as is easily seen from the formula of Theorem 4.3.

While $y_r = 4a \oplus 9ab$ is clearly controllable from a logical viewpoint, it is not controllable from a timing viewpoint, because it would require that the controller delays the uncontrollable event by 1 time unit, which is not allowed. This can be again checked by the formula of Theorem 4.3. Indeed,

$$H_y^{A_u} \circ (H_y^{A_u})^\sharp(4a \oplus 9ab) = 4a \oplus 8ab$$

On the other hand, $y_r = 4a \oplus 8ab$ is already controllable with respect to y and A_u. One might verify that it is indeed a fixpoint of $H_y^{A_u} \circ (H_y^{A_u})^\sharp$, i.e. a controllable series. It corresponds to the supremal controllable series and is given by $y_r = (H_y^{A_u}) \circ (H_y^{A_u})^\sharp(y_{ref})$. A (max,+) automation G_s which realizes y_s, the resulting system, is displayed in figure 1.(c).

5. CONCLUSION

A recently obtained solution to a control problem for (max,+) automata is used in the study of controllability. Controllability of (max,+) formal power series is investigated using residuation theory applied to a generalized Hadamard product of formal power series. Both logical and timing aspects of controllability are characterized within a single formula. Supremal controllable behaviors have been studied. In a future investigation it would be nice to handle unobservable events and to develop decentralized and modular control of concurrent (max,+) automata.

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