Modelling of urban bus networks in dioids algebra

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Abstract. We consider the modelling of urban bus networks in dioid algebras. In particular, we show that their dynamic behavior can be modeled by a Min-Max recursive equation.

1 Introduction

The evolution of a class of Discrete Event Dynamic System (DEDS), *viz* those which involve synchronization phenomena, can be described by linear models provided that a particular algebraic structure, called *dioid* or *idempotent semi-ring*, is used. A linear system theory has been developed by analogy with conventional theory [1, 3]. Applications of this theory have essentially concerned manufacturing systems [8, 6], communication networks [7] and transportation networks [2, 4]. In the latter, the focus has been on systems such as railway networks, which evolve according to timetables. In these systems, synchronization phenomena follow from planned connections and from respect of timetables.

In this paper, we are interested in modelling of urban bus networks whose behaviors differ significantly. In fact, in such systems, synchronizations with timetables occur only at some particular stops (terminus or departure of lines, main stations). In addition, connections between buses are not necessarily planned, but may rather be decided according to various objectives: to absorb peaks of charge in the network, minimize the connection time at intermodal stations, and/or improve the offer of service on strategic itineraries. For those reasons previous models are not appropriate, and we attempt at establishing specific representations for these systems. More precisely, we show that their dynamic behavior can be described by a Min-Max recursive equation. Extending well-known results on fixed-point problems, an 'input/ouput representation' is also deduced.

The outline of the paper is as follows. In §2, we recall elements of dioid theory and principles of DEDS description over dioids. In §3, we study particular fixed-point equations over complete dioids. Their solutions are useful for the modelling of urban bus networks. More precisely, in §4, we first describe how such networks operate in practice, and we next propose their modelling in dioid algebras.

2 Preliminaries

In this section, we give basic notions from the dioid theory and recall succinctly how some DEDS can be modeled in dioid algebras [1, 3].

2.1 Elements of dioid theory

Definition 1. *A dioid is a set $\mathcal{D}$ endowed with two inner operations denoted $\oplus$ and $\otimes$. The sum is associative, commutative, idempotent ($\forall a \in \mathcal{D}$, $a \oplus a = a$) and admits a neutral element denoted ε. The product is associative, distributes over the sum and admits a neutral element denoted e. The element ε is absorbing for the product. A dioid $(\mathcal{D}, \oplus, \otimes)$ is complete if it is closed for infinite sums and if multiplication distributes over infinite sums too.*

Definition 2. *A dioid $(\mathcal{D}, \oplus, \otimes)$ is endowed with a partial order relation denoted $\succeq$ defined by the following equivalence: $a \succeq b \Leftrightarrow a = a \oplus b$.*

A complete dioid has a structure of complete lattice [1, §4.3]. On this account, the greatest lower bound of two elements exists: $a \wedge b = \bigoplus_{\{x \preceq a, x \preceq b\}} x$. Note that $\wedge$ generally distributes over $\oplus$[1], but not over $\otimes$. We only have a *subdistributivity* property of $\otimes$ with respect to $\wedge$: $\forall\, a, b, c \in \mathcal{D}, \quad (a \wedge b)c \preceq ac \wedge bc$. Finally, the following property, called *absorption law*, holds true

$$\forall\, a, b \in \mathcal{D}, \quad a \wedge (a \oplus b) = a \oplus (a \wedge b) = a\,. \tag{1}$$

Example 1 (Dioid $\overline{\mathbb{Z}}_{\max}$). The set $\overline{\mathbb{Z}} = \mathbb{Z} \bigcup \{+\infty, -\infty\}$ endowed with the max operator as sum and the classical sum as product is a complete dioid, usually denoted $\overline{\mathbb{Z}}_{\max}$, with $\varepsilon = -\infty$ and $e = 0$.

Example 2 (Dioid $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$). Let d be a mapping from $\mathbb{Z}$ to $\overline{\mathbb{Z}}_{\max}$. The formal power serie $D(\gamma)$ in one variable γ and coefficients in $\overline{\mathbb{Z}}_{\max}$ is defined by: $D(\gamma) = \bigoplus_{k \in \mathbb{Z}} d(k)\gamma^k$. Let us denote $\langle D(\gamma), \gamma^k \rangle$ the coefficient $d(k)$ of $D(\gamma)$ for γ^k. The set of formal power series in variable γ and coefficients in $\overline{\mathbb{Z}}_{\max}$ endowed with operations $C(\gamma) \oplus D(\gamma) : \langle C(\gamma) \oplus D(\gamma), \gamma^k \rangle = \langle C(\gamma), \gamma^k \rangle \oplus \langle D(\gamma), \gamma^k \rangle$ and $C(\gamma) \otimes D(\gamma) : \langle C(\gamma) \otimes D(\gamma), \gamma^k \rangle = \bigoplus_{i+j=k} \langle C(\gamma), \gamma^i \rangle \otimes \langle D(\gamma), \gamma^j \rangle$ is a dioid denoted $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$.

[1] In all complete dioids considered hereafter, $\wedge$ distributes over $\oplus$. Nevertheless, complete dioids are not necessarily distributive [1, ex. 4.37]

2.2 DEDS description over dioids

It is now well known that the class of discrete event dynamic systems involving only synchronization phenomena can be seen as linear systems over the particular algebraic structure called dioid. For instance, by dating each event, *i.e.* by associating with each event indexed x a *dater*[2] function $\{x(k)\}_{k\in\mathbb{Z}}$, it is possible to get a linear *state representation* in $\overline{\mathbb{Z}}_{\max}$. As in conventional system theory, output $\{y(k)\}_{k\in\mathbb{Z}}$ of a SISO DEDS is then expressed as a convolution of its input $\{u(k)\}_{k\in\mathbb{Z}}$ by its *impulse response* $\{h(k)\}_{k\in\mathbb{Z}}$.
An analogous transform to $\mathcal{Z}$-transform (used to represent discrete-time trajectories in classical theory) can be introduced for daters. Indeed, one can represent a dater $\{x(k)\}_{k\in\mathbb{Z}}$ by its γ-transform which is defined as the following formal power series: $X(\gamma)=\bigoplus_{k\in\mathbb{Z}} x(k)\gamma^k$. Since $\gamma X(\gamma)=\bigoplus_{k\in\mathbb{Z}} x(k)\gamma^{k+1}=\bigoplus_{k\in\mathbb{Z}} x(k-1)\gamma^k$, variable γ can be interpreted as the backward shift operator in event domain. Thus, one can express DEDS behavior over the dioid of formal power series in one variable and coefficients in $\overline{\mathbb{Z}}_{\max}$, denoted $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$[3] (see example 2). In particular, the γ-transform of its impulse response plays the role of *transfer matrix.*

3 fixed-point equations over complete dioids

In this section, we are interested in solving "fixed-point" equations $f(x)=x$, in which f is an isotone (f s.t. $a\preceq b \Rightarrow f(a)\preceq f(b)$) mapping from a complete dioid $\mathcal{D}$ into $\mathcal{D}$. Well known Tarski's theorem[4] states that f admits a least fixed point which coincides with the least solution of inequation $f(x)\preceq x$. Formally, we denote μ_f the least fixed-point of f, then $\mu_f = Inf\{x \mid f(x)\preceq x\}$.

Notation 1 *Let $f:\mathcal{D}\to\mathcal{D}$, we denote $f^0=\mathsf{Id}$, $f^n = f\circ f\circ\ldots\circ f$ (n times) and $f^*=\bigoplus_{n\in\mathbb{N}} f^n$. This 'star notation' applies also for elements $a\in\mathcal{D}$: $a^0=e$, $a^2=a\otimes a$ and $a^*=\bigoplus_{n\in\mathbb{N}} a^n$. Furthermore, we have $a^*=a^*a^*=(a^*)^*$.*

Let us note that the set of fixed point of f^* coincides with the set of *prefix point of f* (x s.t. $f(x)\preceq x$) [1, th. 4.70, p. 186]

$$f(x)\preceq x \Leftrightarrow f^*(x)=x \tag{2}$$

Proposition 1. *Let $\mathcal{D}$ be a complete dioid and $h:\mathcal{D}\to\mathcal{D}$ an isotone mapping. Let $w\in\mathcal{D}$, mapping $g:\mathcal{D}\to\mathcal{D}$ is defined by $g(x)=h(x)\oplus w$. If condition $h(h^*(w))\preceq h^*(w)$ is satisfied, then $\mu_g=h^*(w)$.*

[2] $x(k)$ denotes the $k+1$-th occurence of event x.

[3] Actually, since daters are monotone functions, only a sub-dioid of $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ would be more appropriate to represent γ-transforms of daters (see [1] or [3] for further explanations).

[4] Originally stated for mappings defined over complete lattices, this theorem applies over complete dioids due to their ordered structure (see def. 2).

Proof. According to equivalence (2) we have

$$g(x) = h(x) \oplus w \preceq x \Leftrightarrow h(x) \preceq x \text{ and } w \preceq x \Leftrightarrow h^*(x) = x \text{ and } w \preceq x$$

which implies $h^*(w) \preceq w$. This means that any any prefix point of g, and *a fortiori* μ_g, is greater than $h^*(w)$. Conversely, if $h(h^*(w)) \preceq h^*(w)$ we have

$$g(h^*(w)) = h(h^*(w)) \oplus w \preceq h^*(w) \oplus w = h^*(w)$$

which means that $h^*(w)$ is a prefix point of g and as a by-product $h^*(w) \succeq \mu_g$.

Definition 3. *A mapping $f : \mathcal{D} \to \mathcal{D}$ is said to be lower semi-continuous (l.s.c.) if for every subset $\mathcal{C}$ of $\mathcal{D}$, $f(\bigoplus_{x \in \mathcal{C}} x) = \bigoplus_{x \in \mathcal{C}} f(x)$.*

The following corollary is a well known result (see *e.g.* [1, th. 4.75]).

Corollary 1 *Let $h : \mathcal{D} \to \mathcal{D}$ be a l.s.c. mapping and $g(x) = h(x) \oplus w$, we have $\mu_g = h^*(w)$. In particular, the least fixed point of $g(x) = ax \oplus w$ is $\mu_g = a^*w$.*

Definition 4. *An isotone mapping $f : \mathcal{D} \to \mathcal{D}$ is said to be a closure mapping if $f \succeq \mathsf{Id}$ and $f \circ f = f$.*

If f is a closure mapping, then $f^* = f$ which implies $\forall x, f(f^*(x)) = f(f(x)) = f(x)$. With regard to proposition 1, this leads to the following corollary.

Corollary 2 *Let $h : \mathcal{D} \to \mathcal{D}$ be a closure mapping and $g(x) = h(x) \oplus w$, we have $\mu_g = h^*(w)$. For instance, let $g_1(x) = x^* \oplus w$, we have*[5] *$\mu_{g_1} = w^*$.*

In the next proposition, we present two 'classes of mappings' which are neither l.s.c. nor closure mappings, but for which proposition 1 will even so apply.

Proposition 2. *Let $f : \mathcal{D} \to \mathcal{D}$ be a closure mapping. Mapping $h : \mathcal{D} \to \mathcal{D}$, $h(x) = f(x) \wedge v$ satisfies $h^*(x) = x \oplus (f(x) \wedge v)$ and $h(h^*(x)) \preceq h^*(x)$.*

Proof. If f is a closure mapping $h^2(x) = f(f(x) \wedge v) \wedge v \preceq f(f(x)) \wedge v = f(x) \wedge v$, we then have $h^*(x) = \bigoplus_i h^i(x) = \mathsf{Id} \oplus h(x) = x \oplus (f(x) \wedge v)$. Since $\mathsf{Id} \preceq f$ and using *absorption law* (1), we have $h(h^*(x)) = f(x \oplus (f(x) \wedge v)) \wedge v \preceq f(f(x) \oplus (f(x) \wedge v)) \wedge v = f(f(x)) \wedge v = f(x) \wedge v \preceq h^*(x)$.

The following corollary directly follows from propositions 1 and 2.

Corollary 3 *Let $f : \mathcal{D} \to \mathcal{D}$ be a closure mapping. Let $v, w \in \mathcal{D}$ and $g(x) = (f(x) \wedge v) \oplus w$, we have $\mu_g = (f(w) \wedge v) \oplus w$. For instance, let $g_2(x) = (a^*x \wedge v) \oplus w$ and $g_3(x) = (x^* \wedge v) \oplus w$, we have $\mu_{g_2} = (a^*w \wedge v) \oplus w$ and $\mu_{g_3} = (w^* \wedge v) \oplus w$.*

4 Modelling of public transportation networks

In the following, we are interested in the modelling of urban bus networks. In a first part, we will describe how such networks operate. A model in dioids algebra is proposed in a second part.

[5] Note that generally $(x \oplus y)^* \neq x^* \oplus y^*$, thus corollary 1 cannot apply.

4.1 Exploitation of urban bus networks

As presented in [5, 9], traffic exploitation in urban bus networks can be decomposed in the two following stages.
Definition of an operating schedule The "operating schedule" is established with the aim of optimizing the offer of service according to objectives and exploitation constraints (bus fleet, line layouts, staff hours of work, *etc*). It is calculated for mean conditions of exploitation. In practical terms, this optimization results in:

- the *distribution of resources* throughout the network: number of buses allocated to each line, drivers distribution, *etc.*
- the *synthesis of timetables* defining times at which buses should theoretically run at each stop.

This operating schedule partially conditions the dynamics of the network. In fact, buses are effectively synchronized with timetables at only some stops such as terminus or departures of lines and/or main stations.
Regulation This stage corresponds to adjustments or adaptations from the operating schedule in reaction to current exploitation conditions. Common conditions leading to such adjustment operations are *disturbances*: breakdowns of buses, modifications of traffic flows (for instance due to accidents), *etc.* A supervisor[6] may then decide to transfer passengers, stop or reroute buses... Differently, we are here interested in modelling adjustment operations which rather aim at improving the offer of service by attempting:

1. *to quickly absorb a planned peak of charge in the network.* This operation comes down to postponing buses departures if a sizeable arrival of users is imminent : for instance, near a factory just before closing time, or near a school before home-time...
2. *to provide connections at intermodal stations of the networks.* Such bus stops are located in or near a station where different modes of transport converge (train, subway, tram *etc.*). If an arrival of passengers is imminent, then the operation also consists in waiting for and departing as soon as this quota of users has arrived.
3. *to improve the travelling time on itineraries having priority.* Here, the focus is on itineraries spreading on several bus lines which should be promoted for strategic and/or commercial reasons. With the aim of improving the offer of service on such itineraries, operations then tend to minimize connection times at line changes/switchings.

Let us note that, at a given stop, only one of the above objectives is at most satisfied. In fact, the regulation is *at the earliest*, as specified by the rule below.

Rule 1 *At a given stop, a bus departs as soon as a quota of users has arrived from one of the origins presented at items 1), 2) and 3).*

[6] Visualizing evolutions inside the network and communicating with bus drivers.

4.2 A model for urban bus networks

In this section, we propose a model for urban bus networks operating as described in section 4.1. We assume that such a network includes n bus stops denoted $\mathcal{S}_1, \mathcal{S}_2, \ldots \mathcal{S}_n$. We are interested in the departure times of buses from stops. As the description of traffic exploitation, the modelling issues will be decomposed in two stages: *nominal dynamics* according to the operating schedule, and behaviors induced by operations of regulation.

Nominal dynamics imposed by the operating schedule In the following, let $x_i(k)$ denote the departure time for the $k+1$-st bus from stop $\mathcal{S}_i$. This departure time will be deduced from conditions according to which buses evolve in the network.

We assume, without loss of generality, that at the beginning of operation a bus departs from each stop[7]. A first and obvious condition is that, before departing, buses arrive at stops. Suppose that stop $\mathcal{S}_j$ immediately precedes $\mathcal{S}_i$, then this gives rise to $x_i(k) \succeq a_{ij} + x_j(k-1)$, $k > 1$, in which a_{ij} denotes the travelling time from $\mathcal{S}_j$ to $\mathcal{S}_i$. Let $x(k) = (x_1(k), x_2(k), \ldots x_n(k))^\top$, for the whole network this condition can be written in max-algebraic matrix notation

$$x(k) \succeq A \otimes x(k-1) \tag{3}$$

in which $A_{ij} = \varepsilon$ if $\mathcal{S}_j$ does not precede $\mathcal{S}_i$, otherwise A_{ij} equals to the travelling time from $\mathcal{S}_j$ to $\mathcal{S}_i$.

Another condition is given by the timetable generated for each line. More precisely, at specific stops (see §4.1), buses are synchronized with timetables, that is, they do not depart before the scheduled time. At such a stop $\mathcal{S}_i$, we have $x_i(k) \succeq u_i(k)$, where $u_i(k)$ denotes the scheduled departure time for the $(k+1)$-st bus from $\mathcal{S}_i$. For the whole network, we obtain $x(k) \succeq B \otimes u(k)$ in which $B_{ij} = e$ if $i = j$ and $\mathcal{S}_i$ is a specific stop, $B_{ij} = \varepsilon$ otherwise. Finally, in addition with (3), we get

$$x(k) \succeq Ax(k-1) \oplus Bu(k) \tag{4}$$

Behaviors induced by the regulation operations We assume that peaks of charge described at item 1. are known *a priori* and can consequently be traduced by a vector of daters $\zeta(k)$. Precisely, a coefficient $\zeta_i(k)$ denotes the planned date of arrival at stop $\mathcal{S}_i$ of the k-st quota of users from these flows. In the same manner, we consider that flows of users from others modes of transports are exogenous to our system (see item 2 of §4.1), and we then assume that their dates of occurrence are known *a priori*. In practice, we denote $\rho(k)$ the vector of daters representing dates of arrival at bus stops of quotas of users from other modes of transport.

We consider that several *itineraries having priority* (defined at item 3 of §4.1)

[7] If no or several bus(es) initially depart from stops, then this results only in indexes modifications. These cases can be dealt exactly as cases of places initially containing no or several token(s) for the modelling of timed event graph [1, §2.5.2]

have been selected for the considered urban bus network. Each itinerary is indexed by an element α of the alphabet Σ. Let $\mathcal{S}_i$ be a stop belonging to α, we denote $\xi_i^\alpha(k)$ the date of arrival at $\mathcal{S}_i$ of the k-st quota of passengers following this itinerary. If $\mathcal{S}_j$ precedes $\mathcal{S}_i$ on itinerary α, but does not belong to the same bus line, users have to walk between these stops. We then have $\xi_i^\alpha(k) \succeq f_{ij}^\alpha \otimes \xi_j^\alpha(k)$, $k \geq 1$, in which f_{ij}^α is equal to the connection time between $\mathcal{S}_j$ and $\mathcal{S}_i$ (*e.g.* walking time between these stops), $f_{ij}^\alpha = \varepsilon$ otherwise. For the whole network, this inequality writes

$$\xi^\alpha(k) \succeq F^\alpha \otimes \xi^\alpha(k) \; , \; k \geq 1 \tag{5}$$

with $\xi^\alpha = (\xi_1^\alpha(k), \xi_2^\alpha(k), \ldots, \xi_n^\alpha(k)))^\top$. Differently, if stops $\mathcal{S}_j$ and $\mathcal{S}_i$ follow one another on itinerary α and belong to the same bus line, then we consider that passengers use bus on this portion. We have $\xi_i^\alpha(k) \succeq g_{ij}^\alpha \otimes x_j(k)$, $k \geq 1$, and globally, $\xi^\alpha(k) \succeq G^\alpha \otimes x(k-1)$, $k \geq 1$, in which $G_{ij}^\alpha = A_{ij}$ if $\mathcal{S}_j$ and $\mathcal{S}_i$ follow one another on itinerary α and belong to the same bus line, $G_{ij}^\alpha = \varepsilon$ otherwise. In association with (5), we deduce for itinerary indexed α the following implicit inequation

$$\xi^\alpha(k) \succeq F^\alpha \xi^\alpha(k) \oplus G^\alpha x(k-1) \; , \; k \geq 1 \; .$$

Since we are interested by the earliest functioning of the network, we select the least solution which is given by (corollary 1)

$$\xi^\alpha(k) = F^{\alpha *} G^\alpha x(k-1) \; . \tag{6}$$

Finally, following rule 1, Eq. (6) as well as vectors ζ and ρ can be gathered in an unique inequality representing the influence of regulation operations:

$$x(k) \succeq \bigwedge_{\alpha \in \Sigma} F^{\alpha *} G^\alpha x(k-1) \wedge \zeta(k) \wedge \rho(k) \; . \tag{7}$$

Aggregate model Inequalities (4) and (7) model behaviors induced respectively by the operating schedule and by the regulation operations. Taking into account both aspects leads to

$$x(k) = \Big(\bigwedge_{\alpha \in \Sigma} F^{\alpha *} G^\alpha x(k-1) \wedge \zeta(k) \wedge \rho(k) \Big) \oplus Ax(k-1) \oplus Bu(k) \; .$$

This recurrent equation can be used for the simulation of bus networks. From this 'state equation', we next deduce an input/output representation which should be more suitable to tackle in future works performance evaluation and control of such systems. With that intention, we establish the γ-transform of previous equation using properties[8] $\forall \alpha \in \Sigma \; G^\alpha \preceq A$ and $F^\alpha G^\alpha = F^\alpha A$:

$$x(\gamma) = \Big(\bigwedge_{\alpha \in \Sigma} F^{\alpha *} A \gamma x(\gamma) \wedge \zeta(\gamma) \wedge \xi(\gamma) \Big) \oplus A\gamma x(\gamma) \oplus Bu(\gamma) \; .$$

[8] Deduced from definition of F^α and G^α.

Setting $h(x) = (\bigwedge_{\alpha \in \Sigma} F^{\alpha *} A\gamma x(\gamma) \wedge \zeta(\gamma) \wedge \xi(\gamma)) \oplus A\gamma x(\gamma)$ corollary 1 applies (since h is l.s.c.) to state the least solution $x(\gamma) = h^*(Bu(\gamma))$. To make explicit this solution, we furthermore assume that each itinerary $\alpha \in \Sigma$ includes an unique change of bus-line[9]. We then have ${F^{\alpha}}^2 = \varepsilon$ and $F^\alpha A^i F^\alpha = \varepsilon$, $i \geq 1$. Calculations using notably proposition 1 and corollary 2 lead finally to:

$$x(\gamma) = (A\gamma)^* \Big(\bigwedge_{\alpha \in \Sigma} F^{\alpha *} (A\gamma)^* Bu(\gamma) \wedge \zeta(\gamma) \wedge \xi(\gamma) \Big) \oplus (A\gamma)^* Bu(\gamma).$$

5 Conclusion

This work is a first attempt at modelling dynamic behaviors of urban bus networks in dioids algebra. First of all, we have tried to describe their exploitation, *i.e.*, how they operate in practice. Specificities of such systems have then appeared compared to transportation systems which are governed by timetables (*e.g.* railway networks). We have shown that their dynamic behavior can be described by a Min-Max recurrent equation which can be used for their simulation. An input/ouput representation is also deduced to tackle, in future works, performance evaluation and control of such systems.

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[9] This means that each matrix F^α has only one coefficient different from ε.