Timed Event Graphs with Variable Resources: Asymptotic Behavior, Representation in (min,+) Algebra

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ABSTRACT. *Our aim is to demonstrate that the approach developed for Timed Event Graphs over (min,+) algebra may be extended to a broader subclass of Petri Nets. The considered graphs can be seen as Timed Event Graphs on which some source and/or sink transitions are added to some places.*
Elements of performance evaluation and the linear representation of these systems over the (min,+) algebra (state model with variable parameters and input-output relationship) are proposed.

RÉSUMÉ. *Notre but est de montrer que l'approche développée dans l'algèbre (min,+) pour les graphes d'événements temporisés peut s'étendre à une sous-classe plus large de réseaux de Petri. Les graphes considérés peuvent être vus comme des graphes d'événements temporisés sur lesquels des transitions source et/ou puits sont adjointes à certaines places.*
Des éléments d'évaluation de performance, et la représentation linéaire de ces graphes dans l'algèbre (min,+) (modèle d'état à paramètres variables et relation entrée-sortie) sont donnés.

KEY WORDS : *Timed Petri nets, performance evaluation, (min,+) algebra*

MOTS-CLÉS : *Réseaux de Petri temporisés, évaluation de performance, algèbre (min,+)*

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1. Introduction

Timed Event Graphs (TEG's), which constitute a subclass of timed Petri nets, are well adapted to model Discrete Events Dynamic Systems involving synchronization and saturation phenomena. Their dynamic behaviors can be described by linear equations in a particular algebraic structure called *dioid*, and a system theory has been developed by analogy with conventional linear system theory [BCOQ92], [CMQV89], [MP91]. One can quote results concerning performance evaluation [CMQV89] [Gau95], stabilization [BCOQ92, §6.6] [CHBF99] and control [BCOQ92, §5.6] [BF96]. Our goal is to show that a similar algebraic approach can be extended to a subclass of Timed Free Choice Nets (timed Petri nets for which synchronization phenomena and conflicts are distinct [Mur89], [Gau94]). More precisely, we define a class of timed Petri nets, called *TEG's with variable resources*, which can be likened to linear time-varying systems over $(\min, +)$ *algebra* - a particular dioid. These graphs (TEG's on which source and/or sink transitions are added to certain places) can be modeled by linear equations with variable parameters in $(\min, +)$ algebra and admit an input-output relationship in which the *impulse response* is a bivariate function matrix $h(t, j)$. An entry $[h(t, j)]_{uy}$ is the response at time t of output y resulting from an impulse applied at time j on input u (with no initial energy in the system).

In order to illustrate the considered class of graphs and its modeling power, let us consider the TEG represented in figure 1(a). It models a simple manufacturing system composed of two machines working in parallel and a conveyor. Each part is handled as soon as possible by one of the two machines (processing time equal to τ_1 units of time), and then leaves the workshop on a conveyor (travelling time equal to τ_2 units of time).

In this article, we will be capable of studying for example the system modeled by the graph represented in figure 1(b) which has been built starting from the TEG of the figure 1(a):

- The firing of the additional transitions labeled i_1 and o_1 causes respectively the addition and the withdrawal of one token in the circuit $x_1 \to x_2 \to x_4 \to x_1$, and thus enables to model a variation of the number of machines working in parallel. This variation can, for example, be due to planned maintenance or manufacturing resource scheduling.

- The firing of the additional transitions i_2 and o_2 models respectively the addition and the withdrawal of one part downstream the pool of machines due for example to a conformance test.

The outline of the paper is as follows. In section 2, we recall definitions and notations of the Petri net theory we shall use. The class of graphs, named thereafter *TEG's with variable resources*, is defined in section 3. Elements of performance evaluation of the systems modeled by these graphs are given in section 4. In particular, we study their asymptotic throughputs. Section 5 is devoted to their modeling in $(\min, +)$ algebra. We show that *TEG's with variable resources* can be modeled by $(\min, +)$ linear equations with variable parameters, and specify the input-output relationship.

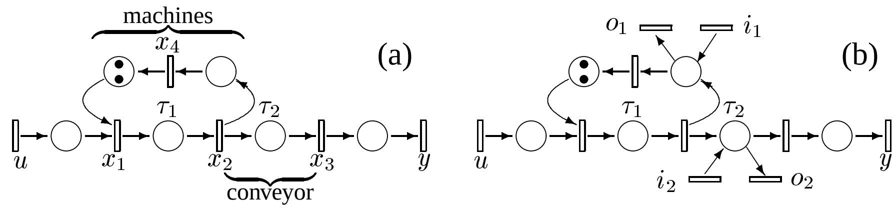

Figure 1. *A TEG (a), a TEG with variable resources (b)*

2. Timed Petri nets

In this section, we introduce definitions and notations of the Petri net theory we shall use throughout the paper (the reader is advised to consult [Mur89] for an exhaustive presentation).

An ordinary *Timed Petri Net* (TPN) is a five-tuple $(\mathcal{P}, \mathcal{Q}, \mathcal{C}, M, \tau)$, in which $\mathcal{P}$ is a finite set of *places*, $\mathcal{Q}$ is a finite set of *transitions*, $\mathcal{C} \subseteq (\mathcal{P} \times \mathcal{Q}) \cup (\mathcal{Q} \times \mathcal{P})$ is a relation between places and transitions, $M \in \mathbb{N}^{\mathcal{P}}$ and $\tau \in \mathbb{N}^{\mathcal{P}}$ are two vectors. The integers M_p and τ_p are called respectively the *initial marking* and the *holding time* of place $p \in \mathcal{P}$. A Petri net is a bipartite graph with two different kinds of nodes, places $p \in \mathcal{P}$ (represented by circles), and transitions $q \in \mathcal{Q}$ (represented by rectangles). An element of $\mathcal{C}$ is an arc from a transition to a place or from a place to a transition. The initial marking M_p is displayed by drawing M_p tokens in place p. A Petri net is a dynamic object, its marking evolves according to the following (earliest) *firing rule*:

1. A transition $q \in \mathcal{Q}$ *fires* as soon as each upstream place contains at least one *available* token.

2. When transition q fires, it consumes one token in each upstream place, and produces one token in each downstream place. A token added in place p at time t becomes *available* at instant $t + \tau_p$.

Let $q \in \mathcal{Q}$, we denote by ${}^\bullet q = \{p \in \mathcal{P} | (p, q) \in \mathcal{C}\}$ (respectively, $q^\bullet = \{p \in \mathcal{P} | (q, p) \in \mathcal{C}\}$) the set of upstream (respectively, downstream) places of q. We define similarly the sets ${}^\bullet p$, $p^\bullet$ as the set of upstream transitions and the set of downstream transitions of place p.

A *timed event graph* (TEG) is a TPN such that each place has exactly one upstream transition and one downstream transition, *i.e.*, $\forall p \in \mathcal{P}, |{}^\bullet p| = |p^\bullet| = 1$.

A *timed free-choice net* (TFCN) is a TPN verifying the following condition

$$\forall p \in \mathcal{P}, q_1, q_2 \in p^\bullet ; \text{ if } q_1 \neq q_2 \text{ then } {}^\bullet q_1 = {}^\bullet q_2 = \{p\}.$$

In other words, if two transitions share an upstream place, they have no other upstream place. TFCN's enable to model systems for which synchronization phenomena (modeled by transitions with several upstream places) and conflicts (modeled by places with several downstream transitions) are distinct. Let us note that a TEG is a TFCN.

3. TEG with variable resources

Definition 1 (TEG with variable resources) *A TEG with variable resources is a TFCN* $\mathcal{G} = (\mathcal{P}, \mathcal{Q} \cup \mathcal{I} \cup \mathcal{O}, \mathcal{C}, M, \tau)$ *satisfying the following conditions:*

1. $\mathcal{G}_{\mathcal{Q}} = (\mathcal{P}, \mathcal{Q}, \mathcal{C}_{\mathcal{Q}}, M, \tau)$ *with* $\mathcal{C}_{\mathcal{Q}} \subseteq (\mathcal{P} \times \mathcal{Q}) \cup (\mathcal{Q} \times \mathcal{P})$ *is a TEG, i.e.,*

$$\forall p \in \mathcal{P}, |\mathcal{Q} \cap p^{\bullet}| = |\mathcal{Q} \cap {}^{\bullet}p| = 1;$$

2. $\mathcal{I}$ *is a set of source transitions, i.e.,* $\forall i \in \mathcal{I}, {}^{\bullet}i = \varnothing$*,*

3. $\mathcal{O}$ *is a set of sink transitions, i.e.,* $\forall o \in \mathcal{O}, o^{\bullet} = \varnothing$*.*

From a structural point of view, such a graph can be seen as a TEG (denoted $\mathcal{G}_{\mathcal{Q}}$) on which source transitions ($\in \mathcal{I}$) and sink transitions ($\in \mathcal{O}$) are connected to some places (see figure 1 for an example). The firings of these source and sink transitions cause respectively addition and withdrawal of resources (tokens) in the underlying TEG ($\mathcal{G}_{\mathcal{Q}}$). These considerations lead to the denomination of *TEG with variable resources*.

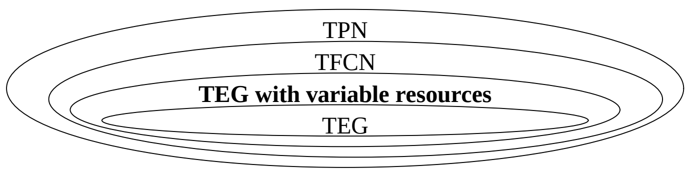

Figure 2. *Venn diagram of timed Petri nets subclasses*

Definition 2 (Counter function) *We associate to each transition* $q \in \mathcal{Q} \cup \mathcal{I} \cup \mathcal{O}$*, a counter function* $q(t)$*, which denotes the cumulated number of firings of transition* q *up to time* t*. We assume that a counter function is defined from* $\mathbb{Z}$ *into* $\mathbb{Z} \cup \{\pm\infty\}$*.*

In the following, counter functions attached to transitions in $\mathcal{I} \cup \mathcal{O}$ are supposed to be known *a priori* (or measured) as exogenous data.

We furthermore assume that any place has at most one upstream (respectively, downstream) transition in $\mathcal{I}$ (respectively, $\mathcal{O}$), *i.e.,*

$$\forall p \in \mathcal{P}, |{}^{\bullet}p \cap \mathcal{I}| \leq 1, \text{ and } |p^{\bullet} \cap \mathcal{O}| \leq 1.$$

This assumption is not restrictive: a set of transitions ($\in \mathcal{I}$) sharing an upstream place p whose collection of counters $\{i(t) | i \in {}^{\bullet}p \cap \mathcal{I}\}$ is given, is equivalent to a unique transition with the given counter $\sum_{i \in {}^{\bullet}p \cap \mathcal{I}} i(t)$.

4. Asymptotic behavior of TEG with variable resources

This section is devoted to performance evaluation of systems modeled by TEG's with variable resources. In a manufacturing systems context, an interesting question is to evaluate the average number of parts the systems can provide by unit of time. Thus, we study here the asymptotic behavior of TEG's with variable resources in order to evaluate the *asymptotic throughput* of their transitions, defined as the average number of firings per unit of time.

Definition 3 (Asymptotic throughput, [Gau95]) *Let $q(t)$ be the counter function of transition q, we denote λ_q its asymptotic throughput defined by:*

$$\lambda_q = \lim_{t \to \infty} \frac{q(t)}{t} .$$

For sake of briefness, we only address the case where the underlying TEG ($\mathcal{G}_{\mathcal{Q}}$) is strongly connected, *i.e.*, $\forall q_1, q_2 \in \mathcal{Q}$, there exists a path from q_1 to q_2 and a path from q_2 to q_1 (cf. remark 3 and reference therein for the non-strongly connected case). It is then possible to evaluate the single asymptotic throughput of transitions in $\mathcal{Q}$ if for each place p_k the asymptotic throughputs of transitions i_k and o_k are equal, *i.e.* , $\lambda_{i_k} = \lambda_{o_k}$.

Let us firstly consider the elementary circuit of a TEG with variable resources represented in figure 3. We denote c this circuit composed of n places and n transitions in $\mathcal{Q}$, which are labeled respectively $p_1, \dots, p_n$ and $x_1, \dots, x_n$.

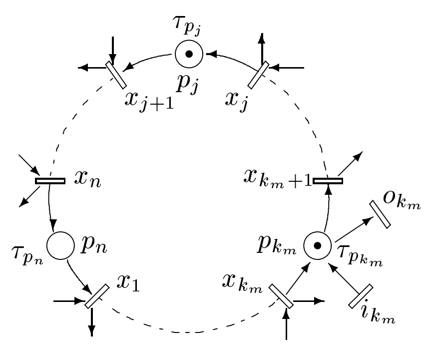

Figure 3. *Elementary circuit c*

The following proposition gives the condition that the asymptotic throughputs of transitions i_k and o_k must satisfy so that all the transitions x_i, $i = 1, \dots, n$ have the same asymptotic throughput.

Proposition 1 *The transitions x_k, $k = 1, \dots, n$, of a circuit c have the same asymptotic throughput if, and only if, $\lambda_{i_k} = \lambda_{o_k}$, $\forall\, k$.*

Proof We suppose that the places $p_{k_1}, \dots, p_{k_r}$ have an upstream transition $i_{k_1}, \dots, i_{k_r} \in \mathcal{I}$ respectively and a downstream transition $o_{k_1}, \dots, o_{k_r} \in \mathcal{O}$ respectively.

• Let us prove that $\lambda_{i_{k_m}} = \lambda_{o_{k_m}}$, $\forall\, m = 1, \dots, r \Longrightarrow \lambda_{x_j} = \lambda_{x_l}$, $\forall\, j, l \in [1, n]$.
Let x_j, x_l ($j \neq l$) be two transitions of circuit c, one notes M_1 (resp. M_2) the finite sum of initial markings of the places belonging to the path from x_j to x_l (resp. from x_l to x_j). It is supposed that $p_{k_1}, \dots, p_{k_{r1}}$ belongs to the path going from x_j to x_l,

and $p_{k_{r1+1}}, \ldots, p_{k_r}$ belongs to the path going from x_l to x_j. We have for all t,

$$\begin{cases} x_l(t) & \leq & x_j(t) + M_1 + \sum_{m=1}^{r1} (i_{k_m}(t) - o_{k_m}(t)) \\ x_j(t) & \leq & x_l(t) + M_2 + \sum_{m=r1+1}^{r} (i_{k_m}(t) - o_{k_m}(t)) \end{cases}$$

hence,

$$x_j(t) - M_2 - \sum_{m=r1+1}^{r} (i_{k_m}(t) - o_{k_m}(t)) \leq x_l(t) \leq x_j(t) + M_1 + \sum_{m=1}^{r1} (i_{k_m}(t) - o_{k_m}(t)),$$

$$\lim_{t\to\infty} \frac{x_j(t) - M_2 - \sum_{m=r1+1}^{r} (i_{k_m}(t) - o_{k_m}(t))}{t} \leq \lim_{t\to\infty} \frac{x_l(t)}{t} \leq$$

$$\lim_{t\to\infty} \frac{x_j(t) + M_1 + \sum_{m=1}^{r1} (i_{k_m}(t) - o_{k_m}(t))}{t}.$$

Since M_1 and M_2 are finite, and $\lim_{t\to\infty} \frac{i_{k_m}(t)}{t} = \lim_{t\to\infty} \frac{o_{k_m}(t)}{t}$, $\forall m = 1, \ldots, r$, we have: $\lambda_{x_j} \leq \lambda_{x_l} \leq \lambda_{x_j}$, or equivalently $\lambda_{x_j} = \lambda_{x_l}$.

• Let us prove that $\lambda_{x_j} = \lambda_{x_l}$, $\forall j, l \in [1, n] \Longrightarrow \lambda_{i_{k_m}} = \lambda_{o_{k_m}}$, $\forall m = 1, \ldots, r$.
Since ${}^\bullet x_{k_m+1} = {}^\bullet o_{k_m} = \{p_{k_m}\}$, $\forall m = 1, \ldots, r$ (the graph is a TFCN), we have with the earliest firing rule (cf. section 2):

$$x_{k_m+1}(t) + o_{k_m}(t) = x_{k_m}(t - \tau_{p_{k_m}}) + i_{k_m}(t - \tau_{p_{k_m}}) + M_{p_{k_m}},$$

$$\lim_{t\to\infty} \frac{x_{k_m+1}(t)}{t} + \lim_{t\to\infty} \frac{o_{k_m}(t)}{t} = \lim_{t\to\infty} \frac{x_{k_m}(t-\tau_{p_{k_m}})}{t} + \lim_{t\to\infty} \frac{i_{k_m}(t-\tau_{p_{k_m}})}{t} + \lim_{t\to\infty} \frac{M_{p_{k_m}}}{t}.$$

Since $M_{p_{k_1}}, \ldots, M_{p_{k_r}}$ are finite and by assumption $\lambda_{x_{k_m+1}} = \lambda_{x_{k_m}}$, $\forall m = 1, \ldots, r$, the preceding equation leads to the result. ■

Remark 1 If at the place labeled p_k only a transition i_k (resp. o_k) exists, the necessary and sufficient condition becomes $\lambda_{i_k} = 0$ (resp. $\lambda_{o_k} = 0$). ⋄

The following proposition gives the maximum value of the asymptotic throughput of transitions x_j, $j = 1, \ldots, n$. To this end, we consider that no transition x_j is synchronized, *i.e.*, $|{}^\bullet x_j| = 1$, so that the evolution of tokens in the circuit c cannot be delayed by the other circuits of the graph.

Let us note $T(c)$, $M(c)$ the sum of holding times and the total number of tokens initially contained in the circuit c. We denote $\overline{q}(t)$ the average value of the counter $q(t)$ up to time t, and $\overline{q}$ its limit when t tends towards the infinity, *i.e.*,

$$\overline{q}(t) = \left(\sum_{j=1}^{t} q(j)\right)/t, \qquad \overline{q} = \lim_{t\to\infty} \overline{q}(t), \qquad t \in \mathbb{N}^*.$$

Proposition 2 *If for $j = 1, \ldots, n$, $\lambda_{i_j} = \lambda_{o_j}$ and $|{}^\bullet x_j| = 1$, then the asymptotic throughput λ_c, identical for all transitions x_j of circuit c, is given by*

$$\lambda_c = \frac{M(c) + \sum_{j=1}^{n} \left(\overline{i_j} - \overline{o_j} - \tau_{p_j} \cdot \lambda_{i_j}\right)}{T(c)}. \tag{1}$$

Proof Let us consider that only the place p_{k_m} has an input (i_{k_m}) and an output (o_{k_m}). We use here the concept of *Resource-Time Product* (RTP) described in [Mur89]. The RTP of a place is the product of the number of tokens (resources) by the length of time that these tokens reside in the place. If one notes $\overline{M_{p_j}}(t)$ the average marking of place p_j up to time t, its RTP up to t is equal to $\overline{M_{p_j}}(t) \cdot t$.

The quantity $\tau_{p_j} \cdot x_j(t - \tau_{p_j})$ is also a product between one duration of residence and a number of tokens. Variable $x_j(t - \tau_{p_j})$ is equal to the number of tokens entered in p_j *via* x_j until instant $t - \tau_{p_j}$; the tokens of initial marking as well as those arrived between instants $t - \tau_{p_j}$ and t are not counted. The residence time of the tokens arriving at p_j is exactly τ_{p_j} units of time since by assumption transition x_{j+1} is not synchronized. In product $\tau_{p_j} \cdot x_j(t - \tau_{p_j})$, the number of tokens present in p_j up to time t is underestimated, which leads to

$$\begin{array}{rl} & \tau_{p_j} \cdot x_j(t - \tau_{p_j}) \leq \overline{M_{p_j}}(t) \cdot t, \quad \text{for } j \neq k_m \\ \text{and} & \tau_{p_{k_m}} \cdot (x_{k_m}(t - \tau_{p_{k_m}}) + i_{k_m}(t - \tau_{p_{k_m}})) \leq \overline{M_{p_{k_m}}}(t) \cdot t. \end{array} \tag{2}$$

On the contrary, in product $\tau_{p_j} \cdot (M_{p_j} + x_j(t))$, where M_{p_j} is the initial marking of p_j, all the tokens present in p_j up to t are counted, but the residence time of some tokens is overestimated. Indeed, the M_{p_j} initial tokens and those arrived between instants $t - \tau_{p_j}$ and t have resided less than τ_{p_j} units of time in p_j, what brings to

$$\begin{array}{rl} & \overline{M_{p_j}}(t) \cdot t \leq \tau_{p_j} \cdot (M_{p_j} + x_j(t)), \quad \text{for } j \neq k_m \\ \text{and} & \overline{M_{p_{k_m}}}(t) \cdot t \leq \tau_{p_{k_m}} \cdot (M_{p_{k_m}} + x_{k_m}(t) + i_{k_m}(t)). \end{array} \tag{3}$$

Summing the inequalities (2) and (3) for each place of the circuit c leads to

$$\sum_{j=1}^{n} \tau_{p_j} x_j(t - \tau_{p_j}) + \tau_{p_{k_m}} i_{k_m}(t - \tau_{p_{k_m}}) \leq t \cdot \left(M(c) + \overline{i_{k_m}}(t) - \overline{o_{k_m}}(t)\right) \leq$$
$$\sum_{j=1}^{n} \tau_{p_j}(M_{p_j} + x_j(t)) + \tau_{p_{k_m}} i_{k_m}(t),$$

while having noted that $\sum_{j=1}^{n} \overline{M_{p_j}}(t)$, namely the average number of tokens contained in circuit c up to t, is equal to $M(c) + \overline{i_{k_m}}(t) - \overline{o_{k_m}}(t)$.
Subtracting $\tau_{p_{k_m}} o_{k_m}(t)$ from both sides and dividing by t, we have

$$\frac{\sum_{j=1}^{n} \tau_{p_j} x_j(t-\tau_{p_j})}{t} + \tau_{p_{k_m}} \frac{i_{k_m}(t-\tau_{p_{k_m}}) - o_{k_m}(t)}{t} \leq M(c) + \overline{i_{k_m}}(t) - \overline{o_{k_m}}(t) - \tau_{p_{k_m}} \frac{o_{k_m}(t)}{t}$$
$$\leq \frac{\sum_{j=1}^{n} \tau_{p_j}(x_j(t) + M_{p_j})}{t} + \tau_{p_{k_m}} \frac{i_{k_m}(t) - o_{k_m}(t)}{t}.$$

With the assumption $\lambda_{i_{k_m}} = \lambda_{o_{k_m}}$, proposition 1 gives $\forall j, l, \lambda_{x_j} = \lambda_{x_l}$, and as $t \to \infty$:

$$\left(\sum_{j=1}^{n} \tau_{p_j}\right)\lambda_{x_j} \leq M(c) + \overline{i_{k_m}} - \overline{o_{k_m}} - \tau_{p_{k_m}} \lambda_{o_{k_m}} \leq \left(\sum_{j=1}^{n} \tau_{p_j}\right)\lambda_{x_j}$$

finally, $$\lambda_c = \lambda_{x_j} = \frac{M(c) + \overline{i_{k_m}} - \overline{o_{k_m}} - \tau_{p_{k_m}} \lambda_{o_{k_m}}}{T(c)}.$$ ■

Let us denote by C the set of elementary circuits of the strongly connected graph $\mathcal{G}_Q$. The asymptotic throughput λ (identical for all the transitions) is

$$\lambda = \min_{c \in \mathrm{C}} \lambda_c, \text{ in which } \lambda_c \text{ is given by equation (1).}$$

Remark 2 By comparison, the asymptotic throughput for strongly connected TEG's is given by the simplified formula [BCOQ92], [Gau95]:

$$\lambda = \min_{c \in \mathrm{C}} \frac{M(c)}{T(c)}.$$

◇

Example 1 Let us consider the elementary circuit represented on figure 4 in which only the place on the right side has a non-zero holding time (1 unit of time). Trajectories of counters $i(t)$ and $o(t)$ are formally defined by:

$$\forall t \in \mathbb{Z}, i(t) = \begin{cases} 0 & t < 0 \\ t+1 & t \geq 0 \end{cases} \text{ and } o(t) = \begin{cases} 0 & t < 4 \\ 2 \cdot \lfloor \frac{t}{2} \rfloor - 2 & t \geq 4 \end{cases},$$

with $\lfloor x \rfloor = \sup\{n \in \mathbb{N} | n \leq x\}$.

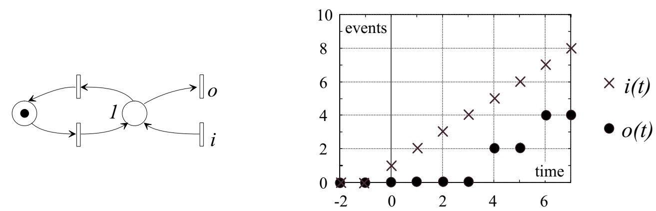

Figure 4. *Example of asymptotic throughput evaluation*

Since $\lambda_i = \lambda_o = 1$, the asymptotic throughput is identical for both transitions and can be evaluated thanks to proposition 2. We leave to the reader the care to check that $\forall t \geq 4, i(t) - o(t) = 3$ (resp. 4) if t is even (resp. odd). Consequently $\sum_{j=1}^{t} i(j) - o(j)$ tends towards $(7/2) \cdot t$ when t tends towards infinity, and as a by-product $\lim_{t \to \infty} \left(\sum_{j=1}^{t} i(j) - o(j) \right) / t = \bar{i} - \bar{o} = 7/2$. We deduce the asymptotic throughput: $\lambda = (1 + 7/2 - 1)/1 = 7/2$.

If the transitions i and o did not exist, the graph would be a TEG and the asymptotic throughput $\lambda = M(c)/T(c)$ would be equal to 1. □

Remark 3 If the graph $\mathcal{G}_Q$ is not strongly connected, the unicity of the asymptotic throughput is not ensured. The calculations of the various asymptotic throughputs require to introduce a partial order relation between the strongly connected components connected by an acyclic directed graph (see [Gau92] for a detailed presentation). ◇

5. Modeling in (min,+) algebra

A TEG can be seen as a linear time-invariant system over $(\min, +)$ algebra (or dually over $(\max, +)$ algebra). This property justified many works relative -in particular- to the performance evaluation, the stabilization and the control of TEG's by analogy with conventional linear system theory [BCOQ92], [CMQV89], [MP91],[BF96], [CHBF99]. We show in this section that the behavior of TEG's with variable resources can also be described by $(\min, +)$ linear equations. Actually, a TEG with variable resources can be likened to a $(\min, +)$ linear time-varying system: we obtain a state model with variable parameters, and an input-output relationship in which the impulse response is a bivariate function matrix.

5.1. *Algebraic preliminaries*

Definition 4 (Dioid) *A dioid $(\mathcal{D}, \oplus, \otimes)$ is a semiring in which the operation $\oplus$ is idempotent (i.e., $\forall a, a \oplus a = a$); neutral elements of $\oplus$, $\otimes$ are respectively noted ε and e.*

In any dioid a *natural order* is defined by:

$a \preceq b \Leftrightarrow a \oplus b = b$ (the least upper bound of $\{a, b\}$ is then equal to $a \oplus b$).

A dioid is *complete* if every subset A of $\mathcal{D}$ admits a least upper bound equal to $\oplus_{x \in A} x$, and if $\otimes$ distributes over finite and infinite sums. The greatest element, noted $\top$, of a complete dioid $\mathcal{D}$ is equal to $\oplus_{x \in \mathcal{D}} x$.

Example 2 ((min,+) algebra) The set $\mathbb{Z} \cup \{\pm\infty\}$ endowed with $\min$ as $\oplus$ and $+$ as $\otimes$ is a complete dioid ($\varepsilon = +\infty$, $e = 0$), and is usually referred to as $(\min, +)$ algebra. Note that the order $\preceq$ in $(\min, +)$ algebra is just reversed with respect to the usual $\leq$. □

Theorem 1 (see [BCOQ92, §4.5.3]) *In a complete dioid, the particular implicit equation*

$$x = a \otimes x \oplus b$$

admits $a^* \otimes b$ as least solution, with $a^* = \bigoplus_{i \geq 0} a^i \quad (a^0 = e,\ a^{i+1} = a^i \otimes a)$.

Example 3 (Matrix dioids) Starting from a "scalar" dioid $\mathcal{D}$, let us consider $p \times p$ matrices with entries in $\mathcal{D}$. The sum and product of matrices are defined conventionally from the sum and product of scalars. This set of matrices endowed with these two operations is also a dioid denoted $\mathcal{D}^{p \times p}$. Furthermore, if $\mathcal{D}$ is complete, $\mathcal{D}^{p \times p}$ is complete too. Note that n-dimensional row or column vector problems can be handled by embedding such vectors in square matrices with $n - 1$ additional arbitrary (identically equal to ε) rows or columns. □

5.2. *Basic evolution equations*

In a TEG with variable resources, a place p may have two downstream transitions. Such a structure is referred to as a *conflict* [Mur89] and exhibits a nondeterminism. In this paper, we use the approach introduced in [Gau94] and called *preselection policy* (or *routing policy*). Briefly, conflicts are then solved thanks to a protocol, an algorithm or a mapping which selects one of the conflicting transitions to fire. The reader can find a general definition in [CGQ95]. In what follows, we consider a particular routing policy referred to as *origin independent* and defined as follows.

Definition 5 *An origin independent routing policy at place p is an integer partition $\left\{\Pi_q^p\right\}_{q\in p^\bullet}$ of its marking. $\Pi_q^p(n)$ expresses the number of tokens reserved for transition q among the n tokens having been present in place p. More formally, Π_q^p is a non-decreasing map defined from $\mathbb{N}$ into $\mathbb{N}$ such that $\forall n,\ \sum_{q\in p^\bullet}\Pi_q^p(n) = n$.*

A routing policy for the net is a collection of routing policies for places.

It is shown in [CGQ95] that counter variables of a TPN with origin independent routing satisfy the following *transition-to-transition* equation:

$$q(t) = \min_{p\in {}^\bullet q}\left[\Pi_q^p\Big(M_{0_p} + \sum_{q'\in {}^\bullet p} q'(t-\tau_p)\Big)\right]. \tag{4}$$

For TEG's with variable resources, let us define the sets $\mathcal{S} = \{p\in\mathcal{P}, |p^\bullet| = 1\}$, and $\mathcal{R} = \{p\in\mathcal{P}, |p^\bullet| = 2\}$ which form a partition[1] of $\mathcal{P}$. According to definition 1, a transition $q \in \mathcal{Q}$ has either several possible upstream places belonging to $\mathcal{S}$, or a single upstream place belonging to $\mathcal{R}$. We show in paragraphs i) and ii) above that in both cases, equation (4) can be written as a $(\min,+)$ linear equation with (a) variable parameter(s).

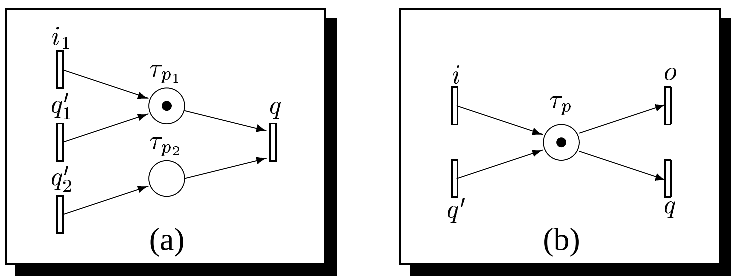

Figure 5. *q downstream to places in $\mathcal{S}$ (a), q and o downstream to a place in $\mathcal{R}$ (b)*

i) $q \in \{q\in\mathcal{Q} | {}^\bullet q \subset \mathcal{S}\}$ (see for example figure 5(a))
Since $\forall p \in {}^\bullet q, |p^\bullet| = 1$, we have $\Pi_q^p = \mathsf{Id}$. In addition, $(\mathcal{P},\mathcal{Q},\mathcal{C}_\mathcal{Q},M,\tau)$ is a TEG; for a given $p\in {}^\bullet q$, the set ${}^\bullet p\cap\mathcal{Q}$ has a single element noted q', and the set ${}^\bullet p\cap\mathcal{I}$ may contain a transition noted i. Equation (4) for counter $q(t)$ can then be written:

$$q(t) = \min_{\{q'\in\mathcal{Q}|q'\in^\bullet({}^\bullet q)\}}\Big[M_p + i(t-\tau_p) + q'(t-\tau_p)\Big],$$

in which $p = q'^\bullet \cap {}^\bullet q$ and with the convention: $i(t) = 0\ \forall t\in\mathbb{Z}$, if ${}^\bullet p\cap\mathcal{I} = \varnothing$. Variable $i(t)$ is supposed to be known at time t (as if it was "gauged"), and we set

[1]According to definition 1 $|p^\bullet \cap \mathcal{Q}| = 1$, and by assumption at §3 $|p^\bullet\cap\mathcal{O}| \le 1$; we then have $|p^\bullet| \in \{1,2\}$.

$\alpha_{qq'}(t,t-\tau_p) = M_p + i(t-\tau_p)$, which can be seen as a time-varying coefficient in the following equation:

$$\begin{aligned} q(t) &= \min_{\{q' \in \mathcal{Q} | q' \in {}^\bullet({}^\bullet q)\}} \left[\alpha_{qq'}(t,t-\tau_p) + q'(t-\tau_p)\right] \\ &= \bigoplus_{\{q' \in \mathcal{Q} | q' \in {}^\bullet({}^\bullet q)\}} \alpha_{qq'}(t,t-\tau_p) \otimes q'(t-\tau_p), \end{aligned} \tag{5}$$

in which $p = q'^\bullet \cap {}^\bullet q$.

ii) $q, o \in \{q \in \mathcal{Q} \cup \mathcal{O} | {}^\bullet q \in \mathcal{R}\}$ (see for example figure 5(b))
By definition, transitions q and o are not synchronized ($|{}^\bullet q| = |{}^\bullet o| = 1$), so the sets ${}^{\bullet\bullet}q \cap \mathcal{Q} = {}^{\bullet\bullet}o \cap \mathcal{Q}$, ${}^{\bullet\bullet}q \cap \mathcal{I} = {}^{\bullet\bullet}o \cap \mathcal{I}$ are singletons composed of transitions respectively noted q' and i. According to equation (4), counters $q(t)$ and $o(t)$ satisfy the following system of equations :

$$\left\{ \begin{array}{lcl} q(t) &=& \Pi_q^p \left(M_p + i(t-\tau_p) + q'(t-\tau_p)\right) \\ o(t) &=& \Pi_o^p \left(M_p + i(t-\tau_p) + q'(t-\tau_p)\right) \end{array} \right.$$

in which $p = q'^\bullet \cap {}^\bullet q = {}^\bullet q = {}^\bullet o$.
By definition, a routing policy does not "consume" token, *i.e.*, at a place p we have $\forall n, \sum_{q \in p^\bullet} \Pi_q^p(n) = n$. In our context, it enables us to write the preceding system in the following form:

$$\left\{ \begin{array}{lcl} q(t) &=& M_p + i(t-\tau_p) - o(t) + q'(t-\tau_p) \\ o(t) &=& \Pi_o^p \left(M_p + i(t-\tau_p) + q'(t-\tau_p)\right). \end{array} \right.$$

Variables $o(t)$ and $i(t)$ are supposed to be known *a priori* at t. Let us set $\alpha_{qq'}(t,t-\tau_p) = M_p + i(t-\tau_p) - o(t)$. Counter $q(t)$ then satisfies the following $(\min,+)$ linear equation with a variable parameter:

$$q(t) = \alpha_{qq'}(t,t-\tau_p) \otimes q'(t-\tau_p), \tag{6}$$

in which $p = q'^\bullet \cap {}^\bullet q = {}^\bullet q = {}^\bullet o$.

In the following, we assume that for $t \leq 0$, $i(t) = o(t) = 0$ which leads to have $\alpha_{qq'}(t,t-\tau_p) = M_p$ for $t \leq 0$ in equations (5) and (6).

Canonical initial conditions

In equations (5) and (6), $\alpha_{qq'}(t,t-\tau_p)$ denotes the sum of initial tokens M_p and of the cumulated number of tokens added in place p (equal to $i(t-\tau_p)$ or $i(t-\tau_p)-o(t)$) up to time t. All these tokens have been available before time t for transition q. Since $\alpha_{qq'}(t,t-\tau_p) = M_p$ for $t \leq 0$, it should be clear that initial tokens have been considered as available since time $-\infty$ to establish equations (5) and (6). This assumption about availability of initial tokens for each place $p \in \mathcal{P}$ is referred to as *canonical initial conditions* [BCOQ92, §2.6.3.4]. One may also write that "the system is with no initial energy", in the sense that all the firings caused exclusively by initial tokens take place at time $-\infty$.

Weakly compatible initial conditions

A classical assumption in Petri net theory is to consider that the graph is "frozen" before time $t = 0$. Tokens "visible" at $t = 0$ are then those of the initial marking. In order to consider such a functioning, we define for each place p a counter $w_p(t)$ which denotes the number of initial tokens which are or have been available for $p^\bullet$ up to time t. We then denote as *weakly compatible initial conditions* [BCOQ92, §2.5.2.1] the following assumptions for each place $p \in \mathcal{P}$:

- initial tokens cannot be available before time 0, *i.e.*, $w_p(t) = 0$ for $t \leq 0$;
- at time τ_p, all the initial tokens have been or are available, *i.e.*, $w_p(\tau_p) \geq M_p$.

In order to obtain the corresponding evolution equations, we define for transition q:

$$v_q(t) = \bigoplus_{p \in {}^\bullet q} w_p(t), \tag{7}$$

and we then have from equations (5) and (6):

$$q(t) = \bigoplus_{q' \in {}^\bullet({}^\bullet q)} \left[\alpha_{qq'}(t, t - \tau_p) \otimes q'(t - \tau_p)\right] \oplus v_q(t), \tag{8}$$

in which $p = q'^\bullet \cap {}^\bullet q$.

Remark 4 In equation (8), $\{v_q(t)\}_{t \in \mathbb{Z}}$ can be seen as the counter function of a fictive transition, located upstream transition q, the role of which is to keep initial tokens in places ${}^\bullet q$ according to the weakly compatible initial condition defined by $\{w_p(t)\}_{t \in \mathbb{Z}}$ for each place $p \in {}^\bullet q$. ◇

5.3. *State representation*

We partition the set of transitions $\mathcal{Q} = \mathcal{U} \cup \mathcal{X} \cup \mathcal{Y}$ in which $\mathcal{U}$ is a set of source transitions, $\mathcal{Y}$ is a set of sink transitions and $\mathcal{X} = \mathcal{Q} \setminus (\mathcal{U} \cup \mathcal{Y})$. We denote by u (resp. x, y) the vector of input (resp. state, output) counters $\{q(t)\}_{t \in \mathbb{Z}}$, $q \in \mathcal{U}$ (resp. $\mathcal{X}$, $\mathcal{Y}$). We denote by τ_m the maximum holding time of a TEG with variable resources, *i.e.*,

$$\tau_m = \max_{p \in \mathcal{P}} [\tau_p].$$

Without loss of generality, we furthermore assume that all the places immediately downstream (resp. upstream) source transitions (resp. sink transitions) have zero holding times (nothing prevents from adding a place with zero holding time and a transition each time this assumption is not satisfied). With the preceding conventions, the dynamic behavior of a TEG with variable resources under canonical initial conditions obeys:

$$\left\{\begin{array}{rcl} x(t) & = & \bigoplus_{i=0}^{\tau_m} A(t,t-i)x(t-i) \oplus B(t,t)u(t) \\ & & \\ y(t) & = & C(t,t)x(t) \end{array}\right. ,t \in \mathbb{Z} \tag{9}$$

in which,

- $A(t,t-i)$ is a $|\mathcal{X}| \times |\mathcal{X}|$ matrix defined by $[A(t,t-i)]_{xx'} = \alpha_{xx'}(t,t-i)$ if there is a place p with $\tau_p = i$ between x' and x, $[A(t,t)]_{xx'} = \varepsilon$ otherwise;

- $B(t,t)$ is a $|\mathcal{X}| \times |\mathcal{U}|$ matrix defined by $[B(t,t)]_{xu} = \alpha_{xu}(t,t)$ if there is a place between u and x, $[B(t,t)]_{xu} = \varepsilon$ otherwise;

- $C(t,t)$ is a $|\mathcal{Y}| \times |\mathcal{X}|$ matrix defined by $[C(t,t)]_{yx} = \alpha_{yx}(t,t)$ if there is a place between x and y, $[C(t,t)]_{yx} = \varepsilon$ otherwise.

The state equation in (9) is implicit due to the places with zero holding times. The dioid composed of matrices with elements in $(\min, +)$ algebra (see examples 2 and 3) being complete, the least solution to this implicit equation exists (cf. theorem 1). It makes sense to select the least solution since it corresponds to the earliest functioning of the net, and the state equation can then be written as follows.

$$x(t) = \bigoplus_{i=1}^{\tau_m} \overline{A}(t,t-i)x(t-i) \oplus \overline{B}(t,t)u(t),\ t \in \mathbb{Z}, \tag{10}$$

with $\overline{A}(t,t-i) = A(t,t)^* \otimes A(t,t-i)$ and $\overline{B}(t,t) = A(t,t)^* \otimes B(t,t)$.

Remark 5 An entry $[A(t,t)^n]_{xx'}$ gives the minimum weight of paths composed of n places with zero holding times from transition x' to transition x. By weight of a path, one should understand here the number of initial tokens plus the cumulated number of tokens "added" (or more precisely, the difference between the counters associated with transitions in $\mathcal{I}$ and those from transitions in $\mathcal{O}$) in places of this path up to time t. As for all t, the weight of a circuit can obviously not be negative, the calculus of $A(t,t)^*$ is finite:

$$A(t,t)^* = \bigoplus_{0\leq i} A(t,t)^i = \bigoplus_{0\leq i\leq |\mathcal{X}|} A(t,t)^i.$$

◇

In order to obtain a recurrence of order 1, we set:

$$\tilde{x}(t) = \begin{pmatrix} x(t) & x(t-1) & \ldots & x(t-\tau_m+1) \end{pmatrix}^T,$$

$$\tilde{u}(t) = u(t),\ \tilde{y}(t) = y(t),$$

$$\tilde{A}(t)=\begin{pmatrix}\overline{A}(t,t-1) & \overline{A}(t,t-2) & \dots & \dots & \overline{A}(t,t-\tau_m)\\ e^{|\mathcal{X}|\times|\mathcal{X}|} & \varepsilon^{|\mathcal{X}|\times|\mathcal{X}|} & \dots & \dots & \varepsilon^{|\mathcal{X}|\times|\mathcal{X}|}\\ \varepsilon^{|\mathcal{X}|\times|\mathcal{X}|} & e^{|\mathcal{X}|\times|\mathcal{X}|} & \varepsilon^{|\mathcal{X}|\times|\mathcal{X}|} & \dots & \varepsilon^{|\mathcal{X}|\times|\mathcal{X}|}\\ \vdots & \ddots & & \varepsilon^{|\mathcal{X}|\times|\mathcal{X}|} & \vdots\\ \varepsilon^{|\mathcal{X}|\times|\mathcal{X}|} & \dots & \varepsilon^{|\mathcal{X}|\times|\mathcal{X}|} & e^{|\mathcal{X}|\times|\mathcal{X}|} & \varepsilon^{|\mathcal{X}|\times|\mathcal{X}|}\end{pmatrix},$$

$$\tilde{B}(t)=\begin{pmatrix}\overline{B}(t,t) & \varepsilon^{|\mathcal{X}|\times|\mathcal{U}|} & \dots & \varepsilon^{|\mathcal{X}|\times|\mathcal{U}|}\end{pmatrix}^{\top},$$

$$\tilde{C}(t)=\begin{pmatrix}C(t,t) & \varepsilon^{|\mathcal{Y}|\times|\mathcal{X}|} & \dots & \varepsilon^{|\mathcal{Y}|\times|\mathcal{X}|}\end{pmatrix}.$$

A TEG with variable resources has finally the following standard state model.

$$\begin{cases}\tilde{x}(t) = \tilde{A}(t)\tilde{x}(t-1)\oplus\tilde{B}(t)\tilde{u}(t)\\ \tilde{y}(t) = \tilde{C}(t)\tilde{x}(t)\end{cases},t\in\mathbb{Z} \qquad (11)$$

Remark 6 By assumption at §5.2, we have $\forall i\in\mathcal{I}$, $\forall o\in\mathcal{O}$, $i(t)=o(t)=0$ for $t\leq 0$, which implies that $\tilde{A}(t)=\tilde{A}(0)$, $\tilde{B}(t)=\tilde{B}(0)$ and $\tilde{C}(t)=\tilde{C}(0)$ for $t\leq 0$. If we furthermore assume that:

$\tilde{x}(t)=\tilde{x}(0)$, $\tilde{u}(t)=\tilde{u}(0)$ and $\tilde{y}(t)=\tilde{y}(0)$ for $t\leq 0$,

the state equation of (11) is implicit for $t\leq 0$, we have:

$$\tilde{x}(0)=\tilde{A}(0)\tilde{x}(0)\oplus\tilde{B}(0)\tilde{u}(0).$$

We select the least solution (cf. theorem 1) which corresponds to the earliest functioning, and equations (11) can then be written:

$$\begin{cases}\tilde{x}(t) = \tilde{A}(0)^{*}\tilde{B}(0)\tilde{u}(0) & ,t\leq 0\\ \tilde{x}(t) = \tilde{A}(t)x(t-1)\oplus\tilde{B}(t)\tilde{u}(t) & ,t>0\\ \tilde{y}(t) = \tilde{C}(t)\tilde{x}(t) & ,t\in\mathbb{Z}.\end{cases}$$

◇

Example 4 We consider the TEG with variable resources represented in figure 1(b), with $\tau_1=1$ and $\tau_2=2$. Its dynamic behavior obeys equations (9) with:

$$x(t)=\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\end{pmatrix}, B(t,t)=\begin{pmatrix}e\\ \cdot\\ \cdot\\ \cdot\end{pmatrix}, A(t,t)=\begin{pmatrix}\cdot & \cdot & \cdot & 2\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & i_1(t)-o_1(t) & \cdot & \cdot\end{pmatrix},$$

$$A(t,t-1)=\begin{pmatrix}\cdot & \cdot & \cdot & \cdot\\ e & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix}, A(t,t-2)=\begin{pmatrix}\cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\\ \cdot & i_2(t-2)-o_2(t) & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix},$$

$$C(t,t)=\begin{pmatrix}\cdot & \cdot & e & \cdot\end{pmatrix},$$

in which each element "$\cdot$" is equal to ε.

We can select the least solution to this implicit equation in order to get equation (10) with

$$[A(t,t)]^* = \begin{pmatrix} e & i_1(t)-o_1(t)+2 & \cdot & 2 \\ \cdot & e & \cdot & \cdot \\ \cdot & \cdot & e & \cdot \\ \cdot & i_1(t)-o_1(t) & \cdot & e \end{pmatrix}, \overline{A}(t,t-1) = \begin{pmatrix} i_1(t)-o_1(t)+2 & \cdot & \cdot & \cdot \\ e & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ i_1(t)-o_1(t) & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\overline{A}(t,t-2) = [A(t,t)]^* \otimes A(t,t-2) = A(t,t-2).$$

The matrices $\tilde{A}(t)$, $\tilde{B}(t)$ and $\tilde{C}(t)$ are easily established (simple rewriting) in order to obtain the standard state model (11). □

Remark 7 (Case of weakly compatible initial conditions) With weakly compatible initial conditions, dynamics of a TEG with variable resources obeys:

$$\left\{ \begin{array}{rcl} x(t) & = & \bigoplus_{i=0}^{\tau_m} A(t,t-i)x(t-i) \oplus B(t,t)u(t) \oplus v_{\mathcal{X}}(t) \\ & & \\ y(t) & = & C(t,t)x(t) \oplus v_{\mathcal{Y}}(t) \end{array} \right. ,t \in \mathbb{Z}, \quad (12)$$

in which,

- $A(t,t-i)$, $B(t,t-i)$ and $C(t,t-i)$ have been defined at equations (9);
- $v_{\mathcal{X}}(t)$ is a $|\mathcal{X}|$ column vector defined by $[v_{\mathcal{X}}(t)]_x = v_x(t)$ (cf. equation (7));
- $v_{\mathcal{Y}}(t)$ is a $|\mathcal{Y}|$ column vector defined by $[v_{\mathcal{Y}}(t)]_y = v_y(t)$ (cf. equation (7)).

With manipulations similar to those described previously, it is possible to obtain the following standard state model:

$$\left\{ \begin{array}{rcl} \tilde{x}(t) & = & \tilde{A}(t)x(t-1) \oplus \tilde{B}(t)u(t) \oplus \tilde{v}_{\mathcal{X}}(t) \\ & & \\ \tilde{y}(t) & = & \tilde{C}(t)\tilde{x}(t) \oplus \tilde{v}_{\mathcal{Y}}(t) \end{array} \right. ,t \in \mathbb{Z} \quad (13)$$

◇

5.4. *Input-output representation*

Starting from the standard state model (11), we will here explicit the input/output relationship and identify the impulse response of TEG's with variable resources.
The state equation of (11) can also be written

$$\tilde{x}(t) = \Phi(t,t_0)\tilde{x}(t_0) \oplus \bigoplus_{j=t_0+1}^{t} \Phi(t,j)\tilde{B}(j)\tilde{u}(j) \ , t \geq t_0$$

in which the *state-transition matrix* $\Phi(t,i)$ is given by

$$\Phi(t,i) = \begin{cases} \text{not defined} & ,i > t \\ Id & ,i = t. \\ \tilde{A}(t) \otimes \tilde{A}(t-1) \otimes \cdots \otimes \tilde{A}(i+1) & ,i < t \end{cases}$$

Then we have, for $t \geq t_0$

$$\tilde{y}(t) = \tilde{C}(t)\Phi(t,t_0)\tilde{x}(t_0) \oplus \bigoplus_{j=t_0+1}^{t} \tilde{C}(t)\Phi(t,j)\tilde{B}(j)\tilde{u}(j). \tag{14}$$

Remark 8 The state-transition matrix satisfies the composition property

$$\Phi(t,j) = \Phi(t,k) \otimes \Phi(k,j) \text{ , with } t \geq k \geq j \text{ ,}$$

and in particular for $t \geq j+1$

$$\Phi(t,j) = \tilde{A}(t)\Phi(t-1,j) = \Phi(t,j+1)\tilde{A}(j+1).$$

◇

Proposition 3 *The least solution of equations (11) is given by*

$$\forall t \in \mathbb{Z}, \quad y(t) = \bigoplus_{j \leq t} h(t,j)\tilde{u}(j) \tag{15}$$

in which h is called the impulse response and is defined by

$$h(t,j) = \tilde{C}(t)\Phi(t,j)\tilde{B}(j), \textit{ for } j \leq t. \tag{16}$$

Proof By tending t_0 towards $-\infty$ in equation (14), it is clear that any solution $\tilde{y}$ is greater than y.
Setting $y(t) = \tilde{C}(t)x(t)$ with

$$x(t) = \bigoplus_{j \leq t} \Phi(t,j)\tilde{B}(j)\tilde{u}(j),$$

we show that x satisfies the state equation in (11):

$$\begin{aligned} x(t) &= \textstyle\bigoplus_{j \leq t} \Phi(t,j)\tilde{B}(j)\tilde{u}(j) \\ &= \textstyle\bigoplus_{j \leq t-1} \Phi(t,j)\tilde{B}(j)\tilde{u}(j) \oplus \tilde{B}(t)\tilde{u}(t) \\ &= \tilde{A}(t)\left[\textstyle\bigoplus_{j \leq t-1} \Phi(t-1,j)\tilde{B}(j)\tilde{u}(j)\right] \oplus \tilde{B}(t)\tilde{u}(t) \quad \text{(thanks to rem. 8)} \\ &= \tilde{A}(t)x(t-1) \oplus \tilde{B}(t)\tilde{u}(t). \end{aligned}$$

■

An element $[h(t,j)]_{yu}$ is the response at time t of output y resulting from an impulse, denoted e_0, at time j and defined by

$$e_0(j) = \begin{cases} e(=0) & ,j \leq 0 \\ \varepsilon(=+\infty) & ,j > 0 \end{cases}$$

applied on input u. Such an input comes down to firing the transition labeled u an infinity of times after time j.

Remark 9 For conventional discrete-time linear time-varying systems [Kam96], the input/output relationship is given by:

$$y(k) = \sum_{j=-\infty}^{k} h(k,j)u(j) \; .$$

The analogy with formula (15) should be clear. ◇

Remark 10 (case of weakly compatible initial conditions) With weakly compatible initial condition, input-output relationship of a TEG with variable resources is given by:

$$y(t) = \bigoplus_{j \leq t} h(t,j)\tilde{u}(j) \oplus y_0(t), t \in \mathbb{Z}, \tag{17}$$

in which

- h is defined by (16);
- y_0 is given by: $y_0(t) = \bigoplus_{j \leq t} \tilde{C}(t)\Phi(t,j)\tilde{v}_{\mathcal{X}}(j) \oplus \tilde{v}_{\mathcal{Y}}(t), t \in \mathbb{Z}.$

◇

6. Conclusion

The studied graphs belong to a subclass of timed free-choice nets which includes the timed event graphs. They can be seen as timed event graphs on which source and/or sink transitions are added to some places. These additional transitions allow modeling additions and withdrawals of resources (tokens) in the course of time.

One gives an expression of the asymptotic throughput if the underlying timed event graph is strongly connected. As for timed event graphs, this expression combines the sum of holding times and the number of tokens initially contained in each circuit, and also the average values of the counters associated with the additional transitions.

Using a *routing policy* (or *preselection policy*), we establish a state model with variable parameters in $(\min,+)$ algebra of systems modeled by these graphs. The parameters, which represent the additions and withdrawals of resources, are supposed to be known *a priori*. Canonical and weakly compatible initial conditions are considered. We also explicit the input-output relationship of these systems. To the sight of these representations, one can say that the studied graphs define a class of $(\min,+)$ linear time-varying systems. The future works will aim at extending to the time-varying case the existing results concerning timed event graphs (which can be seen as linear and time-invariant systems over $(\min, +)$ algebra). A first attempt can be found in [LBH99] in which we generalize the synthesis of the just-in-time control. Furthermore, we are inclined to think that some results for conventional time-varying systems could be adapted to the $(\min, +)$ linear time-varying systems introduced in this paper.

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