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Just In Time Control of Time-Varying Discrete Event Dynamic Systems in $(\max,+)$ Algebra

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We deal with timed event graphs whose holding times associated with places are variable. Defining a First-In-First-Out functioning rule, we show that such graphs can be linearly described in $(\max,+)$ algebra. Moreover, this linear representation allows extending the just in time control synthesis existing for Timed Event Graphs with constant holding times. An example is proposed in order to illustrate how the approach can be applied as a just in time strategy for production lines.

Keywords: Discrete Event Systems, (max,+) Algebra, Time-Varying Systems, Just in Time Control, Timed Event Graphs, Just in Time production, assembly line

1 Introduction

An event graph is a Petri net such that each place has only one input arc and one output arc. Event graphs are well adapted to model synchronization phenomena which can appear in particular in manufacturing systems or in communications networks. If holding times are associated with places, we speak of Timed Event Graphs[1] (TEGs). For example, the k-th token added at time t_k in place p_2 of the TEG represented in figure 1 must stay in this place up to time $t_k + \tau_{p_2}(k)$ before becoming available, and with the earliest functioning rule, transition x_2 fires as soon as a token is available in place p_2. Assuming that, for any place p, the sequence of holding times $\{\tau_p(k)\}_{k\in\mathbb{Z}}$ is such that tokens do not overtake one another when traversing place p, TEGs can be modeled by linear equations in a particular algebraic structure called *dioid*, and a linear system theory has been developed by analogy with conventional system theory (2), (12), (6). Most of the results have been established with constant holding times for which the previous assumption is obviously satisfied. Cofer *et al.* (5) and Brat *et al.* (4) have considered sequences of holding times respectively non-decreasing and periodic such that places are overtake free.

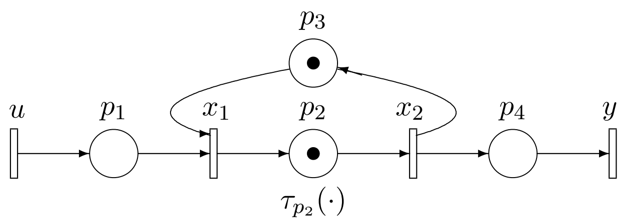

Figure 1. A timed event graph

In this paper, we study systems which can be modeled by TEGs whose functioning is different. Indeed, we define a functioning rule (in practical terms, a condition of availability of tokens) which ensures that places operate as First-In-First-Out (FIFO) channels for all sequences of holding times. For example, the k-th token added in place p_2 at time t_k will have to stay in this place before becoming *available* as long as it is completing its holding time (*i.e.*, up to time $t_k + \tau_{p_2}(k)$) and as long as preceding tokens are

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[1] One can assume without loss of generality that the firings of transitions are instantaneous (2, §2.5.2).

completing their holding times (*i.e.*, up to time $\max_{i<k}[t_i + \tau_{p_2}(i)]$). Such a rule allows studying systems which are "intrinsically" overtake free. Let us consider for example the automobile production line partly shown in figure 2 where cars are handled by an asynchronous linear conveyor crossing successive working areas possibly separated by buffer zones. In each working area, a fixed number of cars can be processed simultaneously (two in the represented working area). Cars cannot overtake one another on the conveyor; buffer zones and working areas work as FIFO channels. The evolution of cars in the considered portion of the line can be modeled by the TEG in figure 1 only if we use the proposed functioning rule. Indeed, a token in place p_2 (representing a car being processed) must not overtake the preceding token(s) even if its holding time $\tau_{p_2}(\cdot)$ (corresponding to the processing time which may depend on the number of optional extras for example) is smaller than those of the preceding token(s).

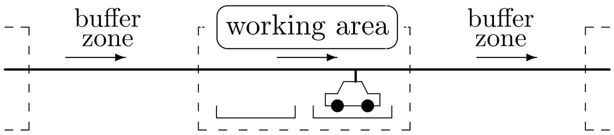

Figure 2. A portion of an automobile production line

TEGs functioning according to the proposed rule are named thereafter *FIFO TEGs*. As tokens cannot overtake one another, we show that these graphs can be modeled by linear equations in the dioid called $(\max, +)$ algebra. This representation constitutes an extension of the modeling approach in $(\max, +)$ algebra.
Assuming that sequences of holding times are given, we present an optimal control of FIFO TEGs under just in time criterion. The synthesis of this control appears to be a generalization of an existing result for TEGs with constant holding times (2, §5.6), (9).
The paper is organized as follows. In Section 2, we define FIFO TEGs and briefly introduce dioids as well as Residuation theory. In Section 3, we give state representation and input-output relationship of FIFO TEGs. In Section 4, the optimal control of FIFO TEGs under just in time criterion is stated. A numerical experimentation of the control synthesis is proposed in Section 5 leading us to a just in time strategy for a car production line.

2 Preliminaries

2.1 *FIFO Timed Event Graphs*

We denote by $\mathcal{P}$ (respectively, $\mathcal{Q}$) the finite set of places (respectively, transitions) of a TEG, and $M_p \in \mathbb{N}$ the number of tokens being initially in place $p \in \mathcal{P}$; $p^\bullet$ (respectively, ${}^\bullet p$) refers to the output transition (respectively, the input transition) of place p. We define similarly sets $q^\bullet$ and ${}^\bullet q$ as the set of output places, and the set of input places, of a transition $q \in \mathcal{Q}$.
The *holding time* is the minimum amount of time that tokens have to stay in a place: the k-th token in place p incurs the holding time denoted $\tau_p(k)$.

Definition 2.1 We define the *earliest FIFO functioning rule* of a TEG as follows.

(i) A transition q *fires* as soon as each place upstream from q contains at least one *available* token.
(ii) We denote $q(n)$ the date of the n-th firing of transition q. This firing consumes one token in each upstream place and produces one token in each downstream place. A token added in place $p \in q^\bullet$ at time $q(n)$ is indexed k with $k = n + M_p$ and becomes available for transition $p^\bullet$ from instant $\max_{1 \le i \le n}\{q(i) + \tau_p(i + M_p)\}$.

A TEG functioning according to this rule is called a *FIFO TEG*.
The only originality in the proposed functioning rule concerns the tokens availability. Classically, the token indexed k in place p is said to be available as soon as its holding time $\tau_p(k)$ is over (7), (2, ch. 2), besides

the above definition of availability ensures that tokens cannot overtake one another when traversing places (places operate as FIFO channels).

2.2 *Dioids, Residuation theory*

Refer to (2) and (3) for exhaustive presentations on dioids and Residuation theory.
A *dioid* is an idempotent semiring ($\mathcal{D}$,$\oplus$,$\otimes$), the neutral elements of operators $\oplus$, $\otimes$ are denoted respectively ε and e. Due to idempotency of $\oplus$, a natural order relation is defined by $a \succeq b \iff a \oplus b = a$ ($a \oplus b$ is the least upper bound of $\{a, b\}$). A dioid $\mathcal{D}$ is *complete* if every subset A of $\mathcal{D}$ admits a least upper bound equal to $\oplus_{\{x \in A\}} x$ and if $\otimes$ distributes over infinite sums (the greatest lower bound of $\{a, b\}$, noted $\wedge$, then exists; $a \wedge b = \oplus_{\{x \preceq a, x \preceq b\}} x$). The greatest element of a complete dioid $\mathcal{D}$ is noted $\top$ and is equal to $\bigoplus_{x \in \mathcal{D}} x$.
Let $\overline{\mathbb{Z}}_{\max}$ be the set $\mathbb{Z} \cup \{\pm\infty\}$ endowed with max as $\oplus$ and usual addition as $\otimes$. It is a complete dioid with neutral elements $\varepsilon = -\infty$ and $e = 0$, and is often called (max,+) algebra.
In this paper, we will also consider the complete dioid denoted (Σ,$\oplus$,$\otimes$) composed of non-decreasing maps from $\mathbb{Z}$ into $\overline{\mathbb{Z}}_{\max}$, and endowed with:

- the pointwise max as $\oplus$, $(u \oplus v)(k) = \max(u(k), v(k))$;
- the *sup-convolution* as multiplication, $(u \otimes v)(k) = \sup\limits_{s \in \mathbb{Z}} \{u(k - s) + v(s)\}$.

The neutral elements are $\varepsilon(k) = -\infty$, $\forall k \in \mathbb{Z}$; $e(k) = -\infty$, $\forall k < 0$ and 0 otherwise.
Starting from a 'scalar' dioid, consider $\mathcal{N} \times \mathcal{N}$ matrices with entries in $\mathcal{D}$. Sum and product of matrices are defined conventionally after the sum and product of scalars in $\mathcal{D}$. This set of matrices endowed with these two operations is a dioid denoted $\mathcal{D}^{\mathcal{N} \times \mathcal{N}}$. The $\mathcal{N}$-dimensional vectors can be handled by embedding such vectors in square matrices with $\mathcal{N} - 1$ additional rows or columns with entries equal to ε (in the following we will 'abusively' speak of dioids of column vectors, then denoted $\mathcal{D}^{\mathcal{N}}$).

A mapping $f : \mathcal{C} \mapsto \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are ordered sets, is *residuated* if for all $y \in \mathcal{D}$, the least upper bound of subset $\{x \in \mathcal{C} \mid f(x) \preceq y\}$ exists and belongs to this subset. It is then denoted $f^\sharp(y)$. Mapping $f^\sharp : \mathcal{D} \mapsto \mathcal{C}$ is called the *residual* of f. When $\mathcal{C}$ and $\mathcal{D}$ are complete dioids, a mapping f is residuated if, and only if, $f(\varepsilon_{\mathcal{C}}) = \varepsilon_{\mathcal{D}}$ and for every subset X of $\mathcal{C}$, $f(\bigoplus_{x \in X} x) = \bigoplus_{x \in X} f(x)$.
In a complete dioid, mapping $x \mapsto a \otimes x$ is residuated; its residual is denoted $y \mapsto a \backslash\!\!\!\circ\, y$ or $y \mapsto \frac{y}{a}$. Some standard formulæ (2, §4.4) which will be useful later on are recalled below :

$$a \otimes (a \backslash\!\!\!\circ\, x) \preceq x \qquad (f.1)$$
$$a \backslash\!\!\!\circ\, (x \wedge y) = (a \backslash\!\!\!\circ\, x) \wedge (a \backslash\!\!\!\circ\, y) \qquad (f.2)$$
$$(a \otimes b) \backslash\!\!\!\circ\, x = \frac{a \backslash\!\!\!\circ\, x}{b} \qquad (f.3)$$

3 Representations of FIFO TEGs in (max,+) algebra

3.1 *State model*

The modeling methodology for FIFO TEGs is similar to the one presented in (2, chap. 2) for TEGs.

Definition 3.1 With each transition $q \in Q$ is associated a *dater variable* also denoted q: $q(k)$ denotes the date of the $(k + 1)$-th firing of transition q. By convention we have $q(k) = +\infty$ if q is fired less than k times, and, $q(k) = \varepsilon$ for all $k < 0$ (*daters* belong to the set Σ of non-decreasing maps from $\mathbb{Z}$ into $\overline{\mathbb{Z}}_{\max}$).

The sequences of holding times $\tau_p(k)$, $p \in \mathcal{P}$, $k \in \mathbb{Z}$ are assumed to be given nonnegative and finite integers.
For sake of briefness, we suppose here that initial tokens are available from instant $-\infty$. Towards the modeling of FIFO TEGs in (max, +) algebra, as for TEGs, this comes down to considering canonical

Figure 3. Illustration of remark *2*

initial conditions (see in (2, §2.5) how to deal with 'non-zero' initial conditions).

Assertion 3.2 The dater variables of a FIFO TEG satisfy the equation

$$q(k) = \bigoplus_{\{q'={}^\bullet p|p\in{}^\bullet q\}} \bigoplus_{i\leq k} \left[\tau_p(i)\otimes q'(i-M_p)\right], \quad k\in\mathbb{Z},$$

or equivalently

$$q(k) = q(k-1)\oplus \bigoplus_{\{q'={}^\bullet p|p\in{}^\bullet q\}} \left[\tau_p(k)\otimes q'(k-M_p)\right], \quad k\in\mathbb{Z}. \tag{1}$$

Remark 1 Eq. (1) can be seen as a particular instance of Eqs. proposed in reference (1) in order to capture queues with customer-varying service times, and to study liveness, absence of explosion as well as stability in integrated service networks.

Remark 2 Eq. (1) does not model the behavior of a TEG (with the standard functioning rule) for which each transition is recycled[1]. Consider for example the simple TEG functioning according to the FIFO rule represented in figure 3.(a). Its dynamic behavior is described by equation

$$q_2(k) = \tau(k)q_1(k)\oplus q_2(k-1) \tag{2}$$

with

k	$q_1(k)$	$\tau(k)$
0	1	3
1	2	1
⋮	⋮	⋮

we have $q_2(0) = 4$ and $q_2(1) = 4$. Eq. (2) does not model the dynamic behavior of the TEG represented in figure 3.(b) functioning according to the usual rule. Indeed, even if transition q_2 is recycled, with the classic rule, tokens may overtake one another in the timed place, and with the previous scenario, the first and second firings of transition q_2 occur at time 3 and 4 respectively.

We partition the set of transitions $\mathcal{Q} = \mathcal{U}\cup\mathcal{X}\cup\mathcal{Y}$, where $\mathcal{U}$ is the set of transitions with no predecessor (input transition), $\mathcal{Y}$ is the set of transitions with no successor (output transition), and $\mathcal{X} = \mathcal{Q}\setminus(\mathcal{U}\cup\mathcal{Y})$ (state transitions). We denote by u (respectively x, y) the vector of input (respectively, state, output) daters q, $q\in\mathcal{U}$ (respectively, $\mathcal{X}$, $\mathcal{Y}$). As for TEGs (see (2, §2.5)), after several manipulations the following *standard state model* can be obtained:

$$\begin{cases} x(k) = A(k)x(k-1)\oplus B(k)u(k) \\ y(k) = C(k)x(k) \end{cases}, k\in\mathbb{Z}, \tag{3}$$

[1]A transition q is said to be recycled if $\{p\in\mathcal{P} \mid p\in{}^\bullet q, p\in q^\bullet, M_p = 1\}\neq\varnothing$.

where $A(k)$, $B(k)$, $C(k)$ are matrices of respective dimensions $\mathcal{N} \times \mathcal{N}$, $\mathcal{N} \times \mathcal{P}$, $\mathcal{R} \times \mathcal{N}$ [2] with entries in $\overline{\mathbb{Z}}_{\max}$.

3.2 *Input-output relationship*

The recursive equation of (3) can also be written

$$x(k) = \Phi(k, k_0)x(k_0) \oplus \bigoplus_{j=k_0+1}^{k} \Phi(k, j)B(j)u(j), k \geq k_0, \tag{4}$$

where the *state-transition matrix* $\Phi(k, i)$ is given by

$$\Phi(k, i) = \begin{cases} \text{not defined} & , i > k \\ e \qquad (\text{identity element of } \overline{\mathbb{Z}}_{\max}{}^{\mathcal{N}\times\mathcal{N}}) & , i = k \\ A(k) \otimes A(k-1) \otimes \cdots \otimes A(i+1) & , i < k \end{cases} .$$

Remark 3 The state-transition matrix satisfies the composition property

$$\Phi(k, i) = \Phi(k, l) \otimes \Phi(l, i) \text{ , where } k \geq l \geq i,$$

and in particular for $k > i$: $\Phi(k, i) = A(k)\Phi(k-1, i)$, $\Phi(k, i) = \Phi(k, i+1)A(i+1)$.

The *input-output relationship* is deduced from Eq. (4) with $x(k_0) = u(k_0) = \varepsilon$ for $k_0 \leq 0$, and is given by

$$\forall k \in \mathbb{Z}, \quad y(k) = \bigoplus_{j \in \mathbb{Z}} h(k, j)u(j) \tag{5}$$

where h is called the *impulse response* and is defined by

$$h(k, j) = \begin{cases} C(k)\Phi(k, j)B(j) & , k \geq j \\ \varepsilon \qquad (\mathcal{Q} \times \mathcal{P} \text{ matrix of null elements of } \overline{\mathbb{Z}}_{\max}) & , k < j \end{cases} . \tag{6}$$

Remark 4 For conventional discrete-time linear time-varying systems (11), (10), the input-output relationship is given by: $y(k) = \sum_{j=-\infty}^{k} h(k, j)u(j)$. The analogy with formula (5) should be clear.

4 Just in time control

The input-output relationship of FIFO TEGs can be written

$$y = \mathcal{H}(u), \tag{7}$$

where the map $\mathcal{H}$: $(\Sigma^{\mathcal{P}}, \oplus, \otimes) \mapsto (\Sigma^{\mathcal{Q}}, \oplus, \otimes)$ is defined by

$$[\mathcal{H}(u)](k) = \bigoplus_{j \in \mathbb{Z}} h(k, j)u(j) \ (cf. \text{ Eq. (5)}).$$

Then the just in time control problem for a FIFO TEG can be formulated as follows: being given the sequences of holding times $\{\tau_p(k)\}_{k \in \mathbb{Z}}$, $p \in \mathcal{P}$ (matrices $\{A(k), B(k), C(k)\}_{k \in \mathbb{Z}}$ and as a by-product the impulse response h are computable), and denoting $z \in \Sigma^{\mathcal{Q}}$ an output signal to be tracked (desired firing

[2] We have $\mathcal{R} = \mid \mathcal{Y} \mid$, but $\mathcal{N}$ (resp. $\mathcal{P}$) may be greater than $\mid \mathcal{X} \mid$ (resp. $\mid \mathcal{U} \mid$) because the state vector x (resp. input vector u) may have been extended to obtain the standard Eqs. (3).

sequences of output transitions), we aim at computing the *greatest control* u (*latest* firing sequences of input transitions) such that

$$y = \mathcal{H}(u) \preceq z,$$

i.e., such that the firing dates of output transitions occur *at the latest before* the desired ones.
This optimal control, denoted u_{opt}, exists if $\mathcal{H}$ is residuated (*cf.* Section 2.2). Dioids $(\Sigma^{\mathcal{P}}, \oplus, \otimes)$ and $(\Sigma^{\mathcal{Q}}, \oplus, \otimes)$ being complete, we only need to show that $\mathcal{H}$ satisfies the necessary and sufficient conditions recalled in Section 2.2:

- $\forall k \in \mathbb{Z}, \quad \big[\mathcal{H}(\varepsilon_{\Sigma^{\mathcal{P}}})\big](k) = \bigoplus_{j\in\mathbb{Z}} h(k,j) \otimes \varepsilon_{\Sigma^{\mathcal{P}}}(j) = \varepsilon_{\Sigma^{\mathcal{Q}}}(k);$
- $\forall k \in \mathbb{Z}, \quad \big[\mathcal{H}(\bigoplus_i u_i)\big](k) = \bigoplus_{j\in\mathbb{Z}} h(k,j) \otimes \big[\bigoplus_i u_i(j)\big] = \bigoplus_i \bigoplus_{j\in\mathbb{Z}} h(k,j) \otimes u_i(j) = \big[\bigoplus_i \mathcal{H}(u_i)\big](k).$

Control input u_{opt} therefore exists and is defined by:

$$u_{opt} \triangleq \bigoplus_{\{u\in\Sigma^{\mathcal{P}} | \mathcal{H}(u)\preceq z\}} u = \mathcal{H}^\sharp(z).$$

Proposition 4.1 *Control $u_{opt}(k)$, $k \in \mathbb{Z}$, is defined by*

$$u_{opt}(k) = \big[\mathcal{H}^\sharp(z)\big](k) = \bigwedge_{i\in\mathbb{Z}} \frac{z(i)}{h(i,k)} \,. \tag{8}$$

Proof Let w be the signal defined by: $\forall k \in \mathbb{Z}$, $w(k) = \bigwedge_{i\in\mathbb{Z}} h(i,k) \backslash\!\!\circ\, z(i)$.

(i) Let us first show that w is a solution to $\mathcal{H}(v) \preceq z$. Using formula $(f.1)$ we have, for all k:

$$\bigoplus_{j\in\mathbb{Z}} h(k,j)w(j) = \bigoplus_{j\in\mathbb{Z}} h(k,j)\Big[\bigwedge_{i\in\mathbb{Z}} \frac{z(i)}{h(i,j)}\Big] \preceq \bigoplus_{j\in\mathbb{Z}} h(k,j)\frac{z(k)}{h(k,j)} \preceq \bigoplus_{j\in\mathbb{Z}} z(k) = z(k).$$

(ii) Let us show that w is greater than any solution to $\mathcal{H}(v) \preceq z$:
$\forall k \in \mathbb{Z}, \bigoplus_{j\in\mathbb{Z}} h(k,j)v(j) \preceq z(k) \Leftrightarrow \forall k \in \mathbb{Z}, \forall j \in \mathbb{Z}; h(k,j)v(j) \preceq z(k)$
$\Leftrightarrow \forall k \in \mathbb{Z}, \forall j \in \mathbb{Z}; v(j) \preceq h(k,j) \backslash\!\!\circ\, z(k) \Leftrightarrow \forall j \in \mathbb{Z}, v(j) \preceq \bigwedge_{k\in\mathbb{Z}} h(k,j) \backslash\!\!\circ\, z(k) = w(j).$

□

In the following, we show that u_{opt} is solution to a system of recurrent equations which proceed backwards in event numbering. These equations offer a strong analogy with the adjoint-state equations of optimal control theory, and constitute an extension to the time-varying case of an existing result for (max,+) linear systems (6), (2, §5.6).

Proposition 4.2 *The control u_{opt} given by Eq. (8) is the greatest solution to*

$$\begin{cases} \xi(k) = \dfrac{\xi(k+1)}{A(k+1)} \wedge \dfrac{z(k)}{C(k)} \\ u(k) = \dfrac{\xi(k)}{B(k)} \end{cases} \quad \forall k \in \mathbb{Z}. \tag{9}$$

Proof Using successively Eq.(6), formulæ (f.3) and (f.2), we have

$$u_{opt}(k) = \bigwedge_{i\in\mathbb{Z}} \frac{z(i)}{h(i,k)} = \bigwedge_{i\geq k} \frac{z(i)}{h(i,k)} = \bigwedge_{i\geq k} \frac{z(i)}{C(i)\Phi(i,k)B(k)} = \frac{\bigwedge_{i\geq k}\left[C(i)\Phi(i,k) \circ\!\!\!\!\backslash\, z(i)\right]}{B(k)}.$$

We set $\overline{\xi}(k) = \bigwedge_{i\geq k} C(i)\Phi(i,k) \circ\!\!\!\!\backslash\, z(i)$.

(i) Let us first show that $\overline{\xi}$ is solution to the first Eq. of (9). We have, for all k:

$$\begin{aligned}
\frac{\overline{\xi}(k+1)}{A(k+1)} \wedge \frac{z(k)}{C(k)} &= \frac{\bigwedge_{i\geq k+1} C(i)\Phi(i,k+1) \circ\!\!\!\!\backslash\, z(i)}{A(k+1)} \wedge \frac{z(k)}{C(k)} \\
&= \bigwedge_{i\geq k+1} \frac{z(i)}{C(i)\Phi(i,k+1)A(k+1)} \wedge \frac{z(k)}{C(k)} && \text{(thanks to } (f.2) \text{ and } (f.3)) \\
&= \bigwedge_{i\geq k+1} \frac{z(i)}{C(i)\Phi(i,k)} \wedge \frac{z(k)}{C(k)\Phi(k,k)} && \text{(owing to Rem. } \mathit{3} \text{ and } \Phi(k,k) = Id) \\
&= \bigwedge_{i\geq k} \frac{z(i)}{C(i)\Phi(i,k)} \\
&= \overline{\xi}(k)\,.
\end{aligned}$$

(ii) Any solution to the first recursive Eq. of (9) can be written

$$\xi(k) = \frac{\xi(k+k_0)}{\Phi(k+k_0,k)} \wedge \bigwedge_{j=k}^{k+k_0-1} \frac{z(j)}{C(j)\Phi(j,k)}, \quad k\in\mathbb{Z}, \quad k_0 \geq 0.$$

We have $\forall k$, $\xi(k) \preceq \overline{\xi}(k)$ when $k_0 \mapsto +\infty$. In other words, $\overline{\xi}$ is greater than any solution to the first Eq. of (9).

□

At the stage of application, an appropriate initial condition of the recursive equation of (9) can be:

$$\exists k_f \text{ such that } z(k) = \top \ (= +\infty) \text{ and } \xi(k) = \top,\ \forall k > k_f.$$

When we are focusing on the control of a manufacturing system, this may mean that either only a finite number (k_f) of products must be delivered, or, the production schedule is not known beyond the k_f-th product to deliver. In both cases, the missing or unknown future orders are then supposed not to constrain the current running of the manufacturing system.

Remark 1 For TEGs with constant holding times, the computation of the difference $\xi_i(k) - x_i(k)$ indicates the greatest delay by which the k-th firing of the transition x_i may be postponed by an exogenous event, with respect to its earliest possible occurrence, without affecting the output transition firing (2, §5.6). In time-varying systems context, a practical information, inspired by the previous notion of 'time margin', is to know if values of some variable holding times can be increased, with respect to their values used to compute the optimal control, without affecting the output transition firing. More precisely, the difference $\xi_i(k) - x_j(k-1)$ indicates the greatest possible value of the holding time $\tau_p(k)$ of place p (with $p \in x_i{}^\bullet$ and $p \in {}^\bullet x_j$) without affecting the output transition firing.

5 Application to a production line

As an example, let us extend the working area of the automobile production line represented in figure 2. In practice, the cars are often carried by a single conveyor along the assembly line to avoid loading and unloading operations between two successive working areas. So, an assembly line is often composed of successive working areas of cars, possibly separated by buffer zones, which are linked with a conveyor system (8), as illustrated in figure 4. Each working area has a particular job to do, *e.g.* placing the windscreens or installing the seat belts. The processing time of a car in a working area may depend on its options and some working areas can process several cars in parallel. We consider here an asynchronous conveyor system, which allows the cars to move forward independently along the assembly line.

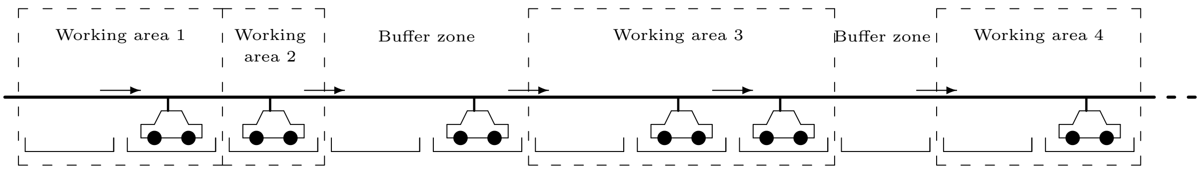

Figure 4. The car assembly line considered as an example

The assembly line represented in figure 4 may be modeled by the FIFO TEG in figure 5. The firing of transition u (respectively, y) modelizes the input (respectively, the output) of a car in production. The firings of internal transitions modelize the inputs and the outputs of cars in the working areas, or in the buffer zones. The holding times in buffer zones, denoted τ_3 and τ_5, have constant values, while the processing times of working areas 1, 2, 3 and 4, denoted $\tau_1(\cdot)$, $\tau_2(\cdot)$, $\tau_4(\cdot)$ and $\tau_6(\cdot)$ respectively, may depend on the indices of tokens. Let us note that the tokens in the timed places represent the position of six cars initially on the assembly line.

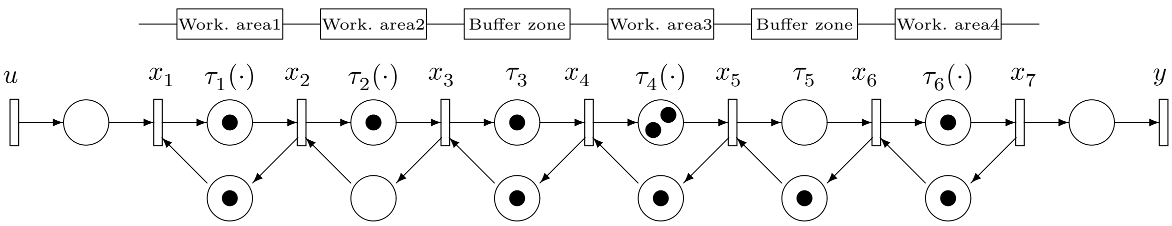

Figure 5. The FIFO TEG corresponding to the system described in figure 4

The state model representing the dynamic behavior of this FIFO TEG (*cf.* Section 3.1) is given by

$$\begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \\ x_7(k) \\ x_4(k-1) \end{pmatrix} = \begin{pmatrix} e & e & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \tau_1(k) & \tau_2(k) & e & e & \cdot & \cdot & \cdot & \cdot \\ \cdot & \tau_2(k) & e & e & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \tau_3 & e & e & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & e & e & \cdot & \tau_4(k) \\ \cdot & \cdot & \cdot & \cdot & \tau_5 & \tau_5 & e & \tau_5 \otimes \tau_4(k) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \tau_6(k) & e & \cdot \\ \cdot & \cdot & \cdot & e & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \otimes \begin{pmatrix} x_1(k-1) \\ x_2(k-1) \\ x_3(k-1) \\ x_4(k-1) \\ x_5(k-1) \\ x_6(k-1) \\ x_7(k-1) \\ x_4(k-2) \end{pmatrix} \oplus \begin{pmatrix} e \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \otimes u(k)$$

$$y(k) = (\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ e\ \cdot) \otimes \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \\ x_7(k) \\ x_4(k-1) \end{pmatrix}$$

in which '·' corresponds to ε.
Four types of cars are supposed to be assembled, their processing times in working areas are given in Table 1.

Type	Area 1	Area 2	Buffer	Area 3	Buufer	Area 4
P_1	1	1	1	2	1	1
P_2	1	2	1	3	1	2
P_3	2	2	1	2	1	2
P_4	2	2	1	3	1	2

Table 1. Processing times in working areas of the four types of cars

Table 2 gives the desired sequence of production for twelve cars (their types and their desired delivery dates). Index k in this sequence starts at 6, since six cars are initially on the assembly line, and will then be previously delivered.

k	6	7	8	9	10	11	12	13	14	15	16	17
Type of cars	P_1	P_2	P_2	P_2	P_2	P_2	P_3	P_3	P_3	P_4	P_4	P_4
$z(k)$	15	16	18	19	22	23	25	26	26	29	30	32

Table 2. Desired sequence of production

Figure 6 represents both the trajectory of the optimal control u_{opt} ("release dates of cars on the assembly line") computed thanks to Eqs. (9), the trajectory of output y ("delivery dates of processed cars") in response to u_{opt} computed with Eqs. (3), as well as the tracked output z ("desired delivery dates of processed cars").

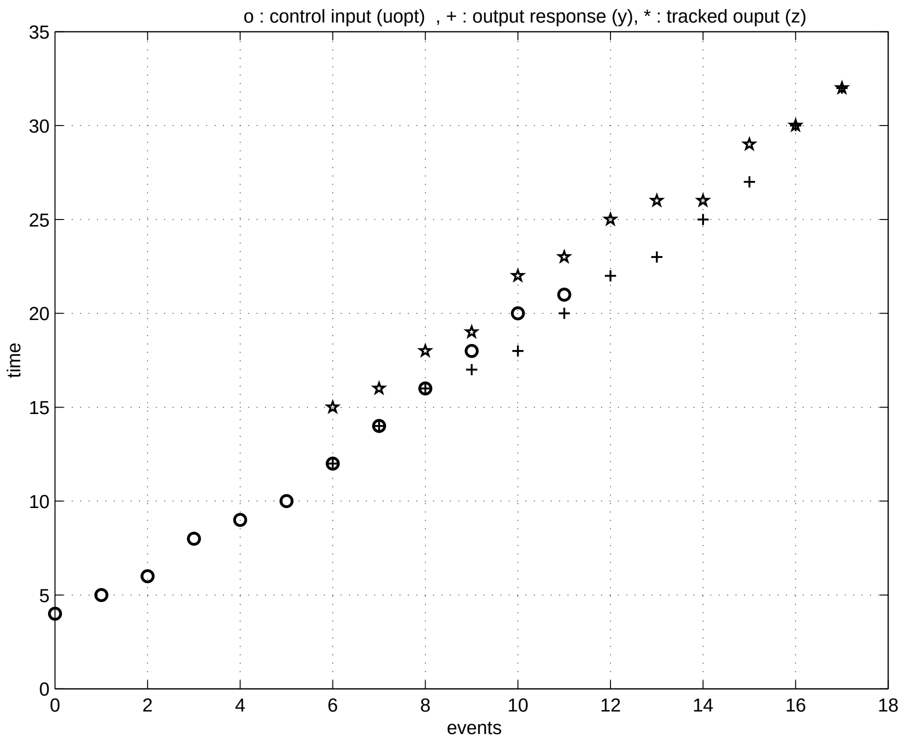

Figure 6. Results for the numerical example

The shift in indices between control input u_{opt} and output response y is due to the six cars initially present on the assembly line.

We observe that output y in response to the computed optimal control u_{opt} is smaller than the tracked output z (given in Table 2). In terms of production system, that means that cars are all delivered before the desired dates. Furthermore, note that the releases of cars in the working area always occur *at the latest* such that processed cars are delivered at or before the desired dates.

6 Conclusion

We have defined a class of timed event graphs whose FIFO characteristic is inherent to a new functioning rule. We give linear representations of these graphs in $(\max, +)$ algebra. The synthesis of the just in time control of such graphs is proposed and is principally based on Residuation theory and dioid properties. Its application to just in time strategy for production lines is presented.

To the sight of their representations, one can say that these graphs allow modelling a class of time-varying $(\max, +)$ linear systems. Possible extensions for this work would then consist in transposing existing results for conventional time-varying linear systems to the max-plus algebraic framework. In particular, authors have studied $(\max, +)$ linear periodic systems in (13) (regarding manufacturing systems, the approach is restricted to cyclic production). A formal definition as well as a characterization of their impulse response have been proposed. Several notions and results from conventional periodic system theory are also transposed to the algebraic context of dioids: the monodromy matrix whose spectral properties are used to study steady states of autonomous systems, the cyclic time invariant reformulation of a periodic state space which is exploited to study state space realization of periodic systems.

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