PERFORMANCE EVALUATION OF DISCRETE EVENT SYSTEMS THANKS TO NEW REPRESENTATIONS FOR (MAX,+) AUTOMATA

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Discrete Event System, performance evaluation, (max,+) automaton.

Abstract:
In this contribution, we study the performances of discrete event systems modeled by (max,+) automata. More precisely, new representations for (max,+) automata are first proposed. From these, several performance indicators can be derived, in particular the maximum time execution and a minorant of the minimum execution time for a sequence of length n. Finally these results are discussed in comparison with several studies of the literature also dealing with performance evaluation of (max,+) automata.

1 INTRODUCTION

At a certain abstraction level, the dynamics of many systems is driven by decisions in reaction to events occurrences. We speak of Discrete Event Systems (DES), and typical examples are manufacturing systems, transportation networks, computer networks (Cassandras and Lafortune, 2008). The motivations can be to identify properties, to analyze and/or to control DES. Different modeling formalisms are used in the literature. In particular, models using (max,+) algebra have been successfully applied to the performance evaluation of DES. Let us mention among others:

- the monograph (Baccelli et al., 1992) for DES which can be modeled by timed event graphs;
- the articles (Gaubert, 1995), (Su and Woeginger, 2011) for DES modeled by (max,+) automata.

The last two works use (max,+) automata as models in order to determine performance indicators, such as the *maximum execution time* and the *minimum execution time* for a sequence of n events, (resp. the maximum and minimum makespan).
In this contribution, the framework is identical, that is, we are interested in performance analysis thanks to models corresponding to automata with weights in (max,+) algebra.
More precisely, recursive equations over (max,+) algebra are proposed to model extremal behaviors of a (max,+) automaton. These constitute representations, which are, to the best of our knowledge, original, and which easily allow us to derive some performance indicators, among which the maximum execution time and a minorant for the minimum execution time.
This paper is organized as follows. In the next section, preliminaries on dioids are recalled together with (max,+) automata and their properties. In Section 3, the new representations for (max,+) automata are introduced. These naturally lead to some performance evaluation elements described in Section 4. A conclusion and some prospects are given in Section 5.

2 PRELIMINARIES

2.1 Dioids

Necessary algebraic concepts on dioids are briefly recalled in this section, see the monographs (Baccelli et al., 1992) and (Heidergott et al., 2006) for an exhaustive presentation.
A *dioid* is a *semiring* in which the addition $\oplus$ is idempotent. The addition (resp, the multiplication $\otimes$) has a unit element ε (resp, e).

Example 1. *The set* $(\mathbb{R}\cup\{-\infty\})$ *with the maximum playing the role of addition and conventional addition playing the role of multiplication is a dioid, denoted* $\mathbb{R}_{\max}$*, with* $e=0$ *and* $\varepsilon=-\infty$*. The set of* $n\times n$ *matrices with coefficients in* $\mathbb{R}_{\max}$*, endowed with the matrix addition and multipli-*

cation conventionally defined from $\oplus$ and $\otimes$, is also a dioid, denoted $\mathbb{R}_{\max}^{n\times n}$. The zero element for the addition is the matrix exclusively composed of ε $(= -\infty)$. We denote I_n the zero element of the multiplication, which is the matrix with e $(= 0)$ on the diagonal and ε $(= -\infty)$ elsewhere.

Example 2. *The set $(\mathbb{R} \cup \{+\infty\})$, with the minimum playing the role of addition and the conventional addition playing the role of multiplication is a dioid, denoted $\mathbb{R}_{min}$ (with $e = 0$ and $\varepsilon = +\infty$), usually called (min,+) algebra.*

Example 3. Formal languages *over a finite alphabet Σ are subsets of free monoid Σ^*, which is composed of finite sequences of letters (called words) from Σ. The set of formal languages, with the union of languages playing the role of addition and concatenation of languages playing the role of multiplication, is a dioid, denoted $(Pwr(\Sigma^*), \cup, .)$. The zero language is $0 = \{\}$, the unit language is denoted $1 = \{\varepsilon\}$ where ε is the empty (zero length) string.*

2.2 (Max,+) automata

Automata with multiplicities in the $\mathbb{R}_{\max}$ semiring are called (max,+) automata. See (Gaubert, 1995) or (Gaubert and Mairesse, 1999) for a more complete introduction.
A (max,+) *automaton* G is a quadruple (Q, Σ, α, μ) where [1]

- Q and Σ are finite sets of states and of events ;
- $\alpha \in \mathbb{R}_{\max}^{1\times|Q|}$ is such that $\alpha_q \neq \varepsilon$ if q is an initial state ;
- $\mu : \Sigma^* \to \mathbb{R}_{\max}^{|Q|\times|Q|}$ is a morphism specified by the matrix family $\mu(a) \in \mathbb{R}_{\max}^{|Q|\times|Q|}$, $a \in \Sigma$, knowing that, for a string $w = a_1 \dots a_n$, we have

$$\mu(w) = \mu(a_1 \dots a_n) = \mu(a_1) \dots \mu(a_n),$$

  where the matrix multiplication involved here, is the one of $\mathbb{R}_{\max}^{|Q|\times|Q|}$. A coefficient $[\mu(a)]_{qq'} \neq \varepsilon$ means that, from state q, the occurrence of event a causes a state transition to q'.

A (max,+) automaton is said to be *deterministic* if

- it has a unique initial state, namely, there is a unique $q \in Q$ such that $\alpha_q \neq \varepsilon$;
- from each state, the occurrence of an event can not induce the occurrence of several possible state transitions, namely, if for all $a \in \Sigma$ each line of $\mu(a)$ contains at most one element not equal to ε.

[1] to simplify the presentation and without loss of generality, the adopted definition omits to distinguish the marked states.

Example 4. *Figure 1 is an example of graphic representation [2] which can be associated with every (max,+) automaton:*

- *the nodes correspond to states $q \in Q$;*
- *an edge exists from state $q \in Q$ to state q' if there exists an event $a \in \Sigma$ such that $[\mu(a)]_{qq'} \neq \varepsilon$: it represents the state transition when event a occurs and the value of $[\mu(a)]_{qq'}$ is interpreted as the duration associated to a (namely, the time activation of event a before it could occur) ;*
- *an input edge symbolizes an initial state.*

For this example, we have $Q = \{I, II\}$, $\Sigma = \{a, b\}$, and

$$\alpha = \begin{pmatrix} e & e \end{pmatrix},\ \mu(a) = \begin{pmatrix} 2 & 3 \\ \varepsilon & \varepsilon \end{pmatrix},$$

$$\mu(b) = \begin{pmatrix} \varepsilon & \varepsilon \\ 6 & 4 \end{pmatrix}.$$

The possible events sequences are the strings: a, b, ab, ba, aa, bb, aab, bba, $aabb$, $abba$, $abab$, ...

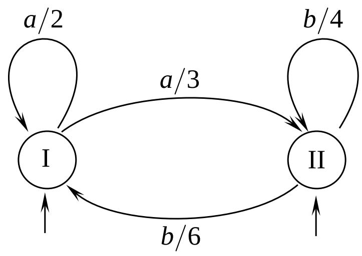

Figure 1: A non deterministic (max,+) automaton.

We define $x_G(w) \in \mathbb{R}_{\max}^{1\times|Q|}$ by

$$x_G(w) = \alpha\mu(w).$$

An element $[x_G(w)]_q$ is interpreted as the date at which the state q is reached consecutively to the events sequence w from an initial state (with the convention that $[x_G(w)]_q = \varepsilon$ if the state q is not

[2] This representation shows that (max,+) automata can be seen as logical automata (like those considered in (Ramadge and Wonham, 1989)) where the time is integrated, namely, as a class of timed automata.

reached consecutively to w). The elements of x_G are *generalized daters*, and we have

$$\begin{cases} x_G(\varepsilon) &= \alpha, \\ x_G(wa) &= x_G(w)\mu(a). \end{cases} \quad (1)$$

3 NEW REPRESENTATIONS FOR (MAX,+) AUTOMATA

Two representations which are to the best of our knowledge original are proposed for deterministic or nondeterministic (max,+) automata. Indeed, the variables associated with (max,+) automata are different from daters considered in (1) since they only account for extremal behaviors.

Let us first introduce several notations.We define the set of triples $H \subset Q \times \Sigma \times Q$ as follows:

$$H = \{(q,a,q') \in Q \times \Sigma \times Q \mid [\mu(a)]_{qq'} \neq \varepsilon\}.$$

A triple (q,a,q') belongs to H if there exists a state transition according to event a from state q to state q'.
For a given event $a \in \Sigma$ and state $q \in Q$, we define the set $H_{a,q} \subset H$ by:

$$H_{a,q} = \{(r,\alpha,s) \in H \mid \alpha = a, s = q\}.$$

We also define the set:

$$\sigma_{n,a,q} = \{[x_G(wa)]_q \mid |w| = n-1\}.$$

Set $\sigma_{n,a,q}$ contains the completion dates for sequences of length n, starting from an initial state, ending with event a and leading to state q, this set is a subset of $\mathbb{R}_{\max}$ and is a *chain* (that is a totally ordered set).
Two representations presented below allow us to determine in particular:

- the maximum element of this subset, that is a performance indicator corresponding to the so-called *worst-case behavior* for the (max,+) automaton;
- and a minorant of this subset, that is a performance indicator related to the so-called *optimal-case behavior* for the (max,+) automaton.

3.1 Representation corresponding to the worst-case behavior

We define the matrix denoted $\overline{A}$ as follows. Let $\overline{A} \in \mathbb{R}_{\max}^{|H|\times|H|}$, and for $j = (p,a,q) \in H$ and $k = (r,a',s) \in H$

$$\overline{A}_{jk} = \begin{cases} [\mu(a)]_{pq} & \text{if } s = p, \\ \varepsilon & \text{otherwise.} \end{cases} \quad (2)$$

Example 5. *The (max,+) automaton represented in figure 1 is such that*

$$H = \{(I,a,I),(I,a,II),(II,b,II),(II,b,I)\},$$

and

$$\overline{A} = \begin{pmatrix} 2 & \varepsilon & \varepsilon & 2 \\ 3 & \varepsilon & \varepsilon & 3 \\ \varepsilon & 4 & 4 & \varepsilon \\ \varepsilon & 6 & 6 & \varepsilon \end{pmatrix}.$$

For example, note that triples (II,b,II) and (I,a,II) are listed respectively as 3rd and 2nd elements in H. Then, $\overline{A}_{3,2} = 4$ brings the information that state transition (II,b,II) can occur consecutively to the occurrence of state transition (I,a,II) with a duration of 4 time units.

Proposition 1. *Let $\overline{x}(n) \in \mathbb{R}_{\max}^{|H|\times 1}$, for $n \in \mathbb{N}$, be defined iteratively by*
$\overline{x}(1)$, with for $j = (p,a,q)$,

$$[\overline{x}(1)]_j = \begin{cases} [\mu(a)]_{pq} & \textit{if } p \textit{ is an initial state}, \\ \varepsilon & \textit{otherwise}, \end{cases} \quad (3)$$

and

$$\overline{x}(n) = \overline{A} \otimes \overline{x}(n-1). \quad (4)$$

Then $\bigoplus_{j\in H_{a,q}} [\overline{x}(n)]_j$ is the maximum element of $\sigma_{n,a,q}$ for each $a \in \Sigma$ and $q \in Q$.

Proof 1. *We use mathematical induction to prove the result. By construction of (3), we have*

$$\bigoplus_{j\in H_{a,q}} [\overline{x}(1)]_j = \bigoplus_{\{p\in Q \mid p \textit{ initial state}\}} [\mu(a)]_{pq},$$

which corresponds to the completion date to reach the state q upon the only occurrence of a (completion date for the sequences of length 1, composed of a and leading to q).

Let us suppose that $\bigoplus_{j\in H_{\alpha,p}} [\overline{x}(n)]_j$ is the maximum element of $\sigma_{n,\alpha,p}$ for all $\alpha \in \Sigma, p \in Q$.
Let us show that for all $a \in \Sigma, p \in Q$, $\bigoplus_{j\in H_{a,p}} [\overline{x}(n+1)]_j$ is the maximum element of $\sigma_{n+1,a,q}$.

$$\begin{aligned} &\textit{We have: } \bigoplus_{j\in H_{a,q}} [\overline{x}(n+1)]_j \\ &= \bigoplus_{j\in H_{a,q}} [\overline{A} \otimes \overline{x}(n)]_j \quad \textit{(using (4))}, \\ &= \bigoplus_{j\in H_{a,q}} \bigoplus_{l\in H} [\overline{A}]_{jl} \otimes [\overline{x}(n)]_l, \\ &= \bigoplus_{l\in H} \bigoplus_{j\in H_{a,q}} [\overline{A}]_{jl} \otimes [\overline{x}(n)]_l, \\ &= \bigoplus_{p\in Q} \bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} \bigoplus_{j\in H_{a,q}} [\overline{A}]_{jk} \otimes [\overline{x}(n)]_k. \end{aligned}$$

Notice that, by definition of (2), we have, for all $k \in H_{\alpha,p}$*,*

$$\bigoplus_{j\in H_{a,q}} [\overline{A}]_{jk} = \begin{cases} \bigoplus_{p'\in Q} [\mu(a)]_{p'q} & \text{if } p = p', \\ \varepsilon & \text{otherwise.} \end{cases}$$

Then, we deduce that

$$\begin{aligned} &\bigoplus_{j\in H_{a,q}} [\overline{x}(n+1)]_j, \\ &= \bigoplus_{p\in Q} \bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} [\mu(a)]_{pq} \otimes [\overline{x}(n)]_k, \\ &= \bigoplus_{p\in Q} [\mu(a)]_{pq} \otimes \Big[\bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} [\overline{x}(n)]_k \Big]. \end{aligned}$$

We know that $\bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} [\overline{x}(n)]_k$ *represents the maximum completion date for sequences of length* n *leading to state* p*, so* $\bigoplus_{j\in H_{a,q}} [\overline{x}(n+1)]_j$ *is the maximum completion date for sequences of length* $n+1$*, ending with event* a *and leading too state* q*.*

Example 6. *Let us consider the non deterministic (max,+) automaton represented in* figure 1. *We have* $Q = \{I, II\}$*,* $\Sigma = \{a,b\}$*, and* $H = \{(I,a,I),(I,a,II),(II,b,II),(II,b,I)\}$*.*

Vector $\overline{x}(n)$ *is written as follows:*

$$\overline{x}(n) = \begin{bmatrix} \overline{x}_{I,a,I}(n) \\ \overline{x}_{I,a,II}(n) \\ \overline{x}_{II,b,II}(n) \\ \overline{x}_{II,b,I}(n) \end{bmatrix}.$$

Its initial value is defined according to (3) by:

$$\overline{x}(1) = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 6 \end{pmatrix}.$$

It satisfies the recursive equation (4), that is:

$$\begin{bmatrix} \overline{x}_{I,a,I}(n) \\ \overline{x}_{I,a,II}(n) \\ \overline{x}_{II,b,II}(n) \\ \overline{x}_{II,b,I}(n) \end{bmatrix} = \begin{pmatrix} 2 & \varepsilon & \varepsilon & 2 \\ 3 & \varepsilon & \varepsilon & 3 \\ \varepsilon & 4 & 4 & \varepsilon \\ \varepsilon & 6 & 6 & \varepsilon \end{pmatrix} \otimes \begin{bmatrix} \overline{x}_{I,a,I}(n-1) \\ \overline{x}_{I,a,II}(n-1) \\ \overline{x}_{II,b,II}(n-1) \\ \overline{x}_{II,b,I}(n-1) \end{bmatrix}.$$

The following table contains the first values obtained thanks to this recurrence in $\mathbb{R}_{\max}$*.*

n	1	2	3	4	5	...
$\overline{x}_{I,a,I}(n)$	2	8	12	17	21	...
$\overline{x}_{I,a,II}(n)$	3	9	13	18	22	...
$\overline{x}_{II,b,II}(n)$	4	8	13	17	22	...
$\overline{x}_{II,b,I}(n)$	6	10	15	19	24	...

For example, the possible sequences of length 3 *ending with event* b *and leading to state* I *are* $\{aab, abb, bbb, bab\}$*. These strings correspond to the following sequences of state transitions*

$I \xrightarrow{a} I \xrightarrow{a} II \xrightarrow{b} I$,

$I \xrightarrow{a} II \xrightarrow{b} II \xrightarrow{b} I$,

$II \xrightarrow{b} II \xrightarrow{b} II \xrightarrow{b} I$,

$II \xrightarrow{b} I \xrightarrow{a} II \xrightarrow{b} I$.

We have $\sigma_{3,b,I} = \{11, 13, 14, 15\}$*.*

On the other hand, we have

$$H_{b,I} = \{(II,b,I)\},$$

hence

$$\bigoplus_{j\in H_{b,I}} [\overline{x}(3)]_j = \overline{x}_{II,b,I}(3) = 15,$$

which corresponds to the maximum element of $\sigma_{3,b,I}$*. In other words,* $\overline{x}_{II,b,I}(3) = 15$ *is the maximum completion date for sequences of length* 3 *ending by event* b *and leading to state* I*.*

3.2 Representation related to the optimal case behavior

In this section, we define a representation in a very similar way to the previous section one, but over (min,+) algebra instead of (max,+) algebra. The reader must have in mind that $\oplus$ then represents the min operation and $\varepsilon = +\infty$.

We define the matrix denoted $\underline{A}$ as follows : $\underline{A} \in \mathbb{R}_{min}^{|H|\times|H|}$ and for $j = (p,a,q) \in H$, $k = (r,a',s) \in H$,

$$\underline{A}_{jk} = \begin{cases} [\mu(a)]_{pq} & \text{if } s = p, \\ \varepsilon & \text{otherwise.} \end{cases} \tag{5}$$

Proposition 2. *Let* $\underline{x}(n) \in \mathbb{R}_{min}^{|H|\times 1}$*, for* $n \in \mathbb{N}$ *be defined iteratively by* $\underline{x}(1)$*, with for* $j = (p,a,q)$*,*

$$[\underline{x}(1)]_j = \begin{cases} [\mu(a)]_{pq} & \text{if } p \text{ is an initial state,} \\ \varepsilon & \text{otherwise.} \end{cases} \tag{6}$$

$$\underline{x}(n) = \underline{A} \otimes \underline{x}(n-1). \tag{7}$$

For all $a \in \Sigma, q \in Q$, $\bigoplus_{j\in H_{a,q}} [\underline{x}(n)]_j$ *is a minorant for* $\sigma_{n,a,q}$*, that is to say, a minorant for the completion dates of sequences of length* n *ending by event* a *and leading to state* q*.*

Proof 2. *The proof goes along the same lines as the proof of proposition 1 and also proceeds by induction. According to (6), we have*

$$\begin{aligned} \bigoplus_{j\in H_{a,q}} [\underline{x}(1)]_j &= \bigoplus_{\{p\in Q | p \text{ initial state}\}} [\mu(a)]_{pq} \\ &= \min_{\{p\in Q | p \text{ initial state}\}} [\mu(a)]_{pq} \end{aligned}$$

which minors the completion date to reach the state q upon the occurrence of the only event a. Let us suppose that for all $\alpha \in \Sigma, p \in Q$, $\bigoplus_{j\in H_{\alpha,p}} [\underline{x}(n)]_j$ is a minorant of $\sigma_{n,\alpha,p}$.
The same arguments as those in the proof of proposition 1 lead to

$$
\begin{aligned}
&\bigoplus_{j\in H_{a,q}} [\underline{x}(n+1)]_j,\\
&= \bigoplus_{p\in Q} [\mu(a)]_{pq} \otimes \left[\bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} [\underline{x}(n)]_k \right].\\
&= \min_{p\in Q} [\mu(a)]_{pq} \otimes \left[\bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} [\underline{x}(n)]_k \right].
\end{aligned}
$$

Since $\bigoplus_{\alpha\in\Sigma} \bigoplus_{k\in H_{\alpha,p}} [\underline{x}(n)]_k$ is a minorant for the completion dates of sequences of length n leading to state p, we can claim that $\bigoplus_{j\in H_{a,q}} [\underline{x}(n+1)]_j$ is also a minorant for the completion dates of sequences of length $n+1$ ending by event a and leading to state q.

Example 7. *Let us consider again the (max,+) automaton represented in figure 1.*
Vector $\underline{x}(n)$ is written as follows:

$$
\underline{x}(n) = \begin{bmatrix} \underline{x}_{I,a,I}(n) \\ \underline{x}_{I,a,II}(n) \\ \underline{x}_{II,b,II}(n) \\ \underline{x}_{II,b,I}(n) \end{bmatrix}.
$$

Its initial value is defined according to(6) by:

$$
\underline{x}(1) = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 6 \end{pmatrix}.
$$

It satisfies the recursive equation (7), that is:

$$
\begin{bmatrix} \underline{x}_{I,a,I}(n) \\ \underline{x}_{I,a,II}(n) \\ \underline{x}_{II,b,II}(n) \\ \underline{x}_{II,b,I}(n) \end{bmatrix} = \begin{pmatrix} 2 & \varepsilon & \varepsilon & 2 \\ 3 & \varepsilon & \varepsilon & 3 \\ \varepsilon & 4 & 4 & \varepsilon \\ \varepsilon & 6 & 6 & \varepsilon \end{pmatrix} \otimes \begin{bmatrix} \underline{x}_{I,a,I}(n-1) \\ \underline{x}_{I,a,II}(n-1) \\ \underline{x}_{II,b,II}(n-1) \\ \underline{x}_{II,b,I}(n-1) \end{bmatrix}.
$$

The following table contains the first values obtained thanks to this recurrence in $\mathbb{R}_{min}$:

n	1	2	3	4	5	...
$\underline{x}_{I,a,I}(n)$	2	4	6	8	10	...
$\underline{x}_{I,a,II}(n)$	3	5	7	9	11	...
$\underline{x}_{II,b,II}(n)$	4	7	9	11	13	...
$\underline{x}_{II,b,I}(n)$	6	9	11	13	15	...

We mentioned in the last example that $\sigma_{3,b,I}$, that is the set of the possible completion dates for the sequences of length 3 ending by event b and leading to state I, is given by

$$\sigma_{3,b,I} = \{11, 13, 14, 15\},$$

and $H_{b,I} = \{(II, b, I)\}$. We then have

$$\bigoplus_{j\in H_{b,I}} [\underline{x}(3)]_j = \underline{x}_{II,b,I}(3) = 11.$$

For this example, $\bigoplus_{j\in H_{b,I}} [\underline{x}(3)]_j$ is not only a minorant for $\sigma_{3,b,I}$ (in accordance with the proposition), but it is also the minimum element of this set.

Let us now mention a case where the obtained minorant is not a minimum element. The set of the possible sequences of length 2 ending by event a and leading to state I is: $\{ba, aa\}$.
Notice that string aa can be generated upon the two paths $I \xrightarrow{a} I \xrightarrow{a} II$, $I \xrightarrow{a} I \xrightarrow{a} I$, and we have $[x_G(aa)]_1 = 5$ (weight of the first path). This is because the definition of the dater associated to the (max,+) automaton holds the maximum completion date for sequence aa (the second path is not recognized by the (max,+) automaton). We then have $\sigma_{2,a,I} = \{5, 8\}$.
We also have $H_{a,I} = \{(I, a, I)\}$ and according to the table below:

$$\bigoplus_{j\in H_{a,I}} [\underline{x}(2)]_j = \underline{x}_{I,a,I}(2) = 4.$$

The result $\bigoplus_{j\in H_{a,I}} [\underline{x}(2)]_j$ does not belong to $\sigma_{2,a,I}$, so it is only a minorant of this set (and not a minimum element).

4 CONTRIBUTIONS TO THE PERFORMANCE EVALUATIONS OF DES

In what follows, we highlight some performance evaluation elements that are provided by the representations proposed in the previous section. We only focus on maximum and minimum execution time for a sequence of given length, which have previously been studied in the literature on (max,+) automata (Gaubert, 1995), (Su and Woeginger, 2011), (Gaubert and Mairesse, 1999).

4.1 Maximum execution time for sequences of given length

For some systems, it is important to have knowledge of the maximum execution time for the se-

quences of given length n.
Its calculation is presented in (Gaubert, 1995) as follows:

$$l_n^{worst} = \bigoplus_{w\in\Sigma^n}\bigoplus_{p\in Q}[x_G(w)]_p, = \bigoplus_{w\in\Sigma^n}\bigoplus_{p\in Q}[\alpha\mu(w_1)...\mu(w_n)], = \bigoplus_{p\in Q}[\alpha M^n]_p,$$

with $M = \bigoplus_{a\in\Sigma}\mu(a)$.
Another computation method, using *heap models*), is presented in (Su and Woeginger, 2011). In both cases the algorithmic complexity is polynomial (but lower with the second method).
The representation given in Proposition 1, allows this indicator to be evaluated (with a polynomial complexity also since it only implies multiplications of matrices over $\mathbb{R}_{\max}$) as:

$$l_n^{worst} = \bigoplus_{j\in H}[\overline{x}(n)]_j.$$

Note that it is possible to refine the indicator, by calculating the maximum completion date for a sequence of length n leading to a specific state q and ending by a given event a:

$$\bigoplus_{j\in H_{a,q}}[\overline{x}(n)]_j = \bigoplus_{j\in H_{a,q}}[\overline{A}^{n-1}\overline{x}(1)]_j.$$

4.2 Minimum execution time for sequences of given length

In some cases, it may be important to have the knowledge of the minimum execution time for sequences of given length n. It is presented in (Gaubert, 1995) as follows:

$$l_n^{opt} = \bigoplus_{w\in\Sigma^n}\bigoplus_{p\in Q}[x_G(w)]_p,$$

where $\bigoplus$ represents the min operation.
Its calculation is only presented for a reduced class of (max,+) automata (deterministic automata), and the algorithmic complexity is high. In (Su and Woeginger, 2011), it is shown that the general case is NP-complete.
In Proposition 2 and thanks to the new representation, we can easily approximate this indicator for all (max,+) automata (knowing that we can sometimes get the exact value (see example 7)), as

$$l_n^{opt} \geq \bigoplus_{j\in H}[\underline{x}(n)]_j.$$

A refinement is also possible for specific final state and event:

$$\bigoplus_{j\in H_{a,q}}[\underline{x}(n)]_j = \bigoplus_{j\in H_{a,q}}[\underline{A}^{n-1}\underline{x}(1)]_j.$$

5 CONCLUSIONS

We have proposed new representations for (max,+) automata. We have shown that these could be applied to performance evaluation and notably to get: the *maximum execution time for sequences of length* n and *a minorant for the minimum execution time for sequences of length* n.
In future investigations, the exact algorithmic complexity of the method should be determined and compared with the existing methods. Spectral properties of matrices $\overline{A}$ and $\underline{A}$ should be exploited to reduce this complexity. We also think that additional performance indicators could be derived from the proposed representations for (max,+) automata.
A control approach, inspired by that for logical automata presented in (Ramadge and Wonham, 1989), has been proposed for (max,+) automata in (Komenda et al., 2009). The proposed representations could be used to elaborate alternative control laws. Since the representations are similar to standard state-space representations, we should consider to transpose the control laws developed for linear (max,+) and (min,+) systems, for example in (Houssin et al., 2007) and (Lahaye et al., 1999).

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