Linear Periodic Systems Over Dioids

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Abstract. A specification of the linear system theory over dioids is proposed for periodic systems. Using the conventional periodic system theory as a guideline, we study periodic systems for which the underlying algebraic structure is a dioid. The focus is on representations (impulse response and state model) associated with such systems, the properties of these representations as well as the state space realization.

Keywords: discrete event systems, linear periodic systems, dioids, state space realization

1. Introduction

Linear systems hold a very important place in systems theory. Besides, results on linear systems are so wealthy that control engineers very often attempt to analyze non-linear systems *via* a linear approximation. This approach is not always possible, and in particular, *Discrete Event Dynamic Systems* (DEDS) cannot reasonably be approximated by usual linear models. In fact, basic phenomena that characterize their dynamics, such as *synchronization* and *competition*, are very nonlinear and nonsmooth phenomena. Typical examples of DEDS are flexible manufacturing systems, telecommunication networks, parallel processing systems and logistic systems, the importance of which is constantly increasing with the emergence of new technologies. Nevertheless, it has been shown that a class of DEDS, those involving synchronization phenomena, can be modeled by linear equations in particular algebraic structures, called *dioids*. For about twenty years, this property has motivated the elaboration of a "new linear system theory" in which the underlying algebra is a dioid. This theory offers a striking analogy with conventional linear system theory since concepts such as state representation, transfer matrices, optimal control, corrector synthesis and identification theory have been introduced (Cuninghame-Green, 1979; Cohen et al., 1989; Baccelli et al., 1992; Cottenceau et al., 1999; Menguy et al., 2000).
At the current stage, studies essentially concern linear time-invariant systems[1]. Nevertheless, lots of systems arising in practice are time-varying, that is, the values of their output response depend on when the input is applied. Time variation is a result of system parameters changing which may happen for example in a manufacturing system, when processing times of successive parts are different. Starting from this observation, linear time-varying systems over dioids have been studied in (Lahaye et al., 1999a; Lahaye et al., 1999b; Lahaye, 2000) with the aim of widening the field of application of the linear systems theory over dioids. In (Lahaye, 2000), among relevant

[1] Their behavior is usually graphically represented by timed event graphs with constant timings (see (Cohen et al., 1989)).

applications, the so-called *repetitive manufacturing systems* have been identified as linear systems whose parameters admit periodic variations. In this paper, the focus is precisely on this particular class of linear time varying systems, namely the *linear periodic systems*. These systems have received much attention in conventional system theory (see *e.g.* (Bittanti, 1996) and references therein), and, using these results as a guideline, we aim at sketching in this paper a specific analysis for linear periodic systems over dioids.
The outline of the paper is as follows. In section 2, some elements of the linear system theory over dioids are presented. Linear periodic systems are introduced in section 3. Starting from a formal definition, several properties of representations associated with such systems are exhibited. Section 4 is devoted to the state space realization problem for linear periodic systems. We first introduce a time-invariant reformulation of a periodic state space representation. From this so-called *cyclic reformulation* we establish a necessary and sufficient condition for the existence of a periodic realization of the impulse response of a periodic system.

2. Preliminaries

2.1. Dioid structure

DEFINITION 1. A dioid is a set $\mathcal{D}$ endowed with two inner operations denoted $\oplus$ and $\otimes$[2]. The sum is associative, commutative, idempotent ($\forall a \in \mathcal{D}$, $a \oplus a = a$) and admits a neutral element denoted ε. The product is associative, distributes over the sum and admits a neutral element denoted e. Element ε is absorbing for the product.

In any dioid, a natural order is defined by:

$$a \preceq b \Leftrightarrow a \oplus b = b \quad (a \oplus b \text{ is the least upper bound of } \{a, b\}).$$

DEFINITION 2. A dioid $\mathcal{D}$ is complete if each subset A of $\mathcal{D}$ admits a least upper bound denoted $\bigoplus_{x \in A} x = \sup_{x \in A} x$, and if $\otimes$ distributes with respect to infinite sums. In particular, $\top = \bigoplus_{x \in \mathcal{D}} x$ is the greatest element of $\mathcal{D}$.

EXAMPLE 3. Let $\overline{\mathbb{R}}_{\max}$ be the set $\mathbb{R} \cup \{\pm\infty\}$ endowed with max as $\oplus$ and usual addition as $\otimes$. It is a complete commutative dioid with neutral elements $\varepsilon = -\infty$ and $e = 0$ ($\top = +\infty$).

DEFINITION 4. [see (Gaubert, 1994, def. 1.1.5)] A dioid $\mathcal{D}$ satisfies the weak stabilization condition if for all $a, b, \lambda, \mu \in \mathcal{D}$, there exist $c, \nu \in \mathcal{D}$ and $N \in \mathbb{N}$ such that

$$n \geq N \;\Rightarrow a\lambda^n \oplus b\mu^n = c\nu^n \,.$$

EXAMPLE 5. Dioid $\overline{\mathbb{R}}_{\max}$ satisfies the weak stabilization condition.

[2] As usual, the multiplicative sign may sometimes be omitted.

In the following, we shall consider vectors and matrices with entries in a dioid. The sum and product of matrices are defined conventionally. Let $A, B \in \mathcal{D}^{n\times n}$,

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}\ , \quad (A \otimes B)_{ij} = \bigoplus_{l=1}^{n} A_{il} \otimes B_{lj}\ .$$

The matrix-vector product is defined in a similar way. We denote Id_n the $n \times n$ identity matrix with entries equal to e on the diagonal and to ε elsewhere.
Given a matrix $M \in \mathcal{D}^{n\times n}$, we shall also consider the problem of existence of eigenvalues and eigenvectors in $\mathcal{D}$, that is, the existence of (non-ε) $\lambda \in \mathcal{D}$ and $v \in \mathcal{D}^n$ such that:

$$M \otimes v = \lambda \otimes v.$$

This spectral problem has been extensively studied. We only recall basic definitions and results, the reader can consult (Baccelli et al., 1992; Gaubert, 1992; Braker, 1993) for exhaustive presentations.

DEFINITION 6. A matrix $M \in \mathcal{D}^{n\times n}$ is irreducible if $\forall i, j$; $\exists l \geq 0$ s.t. $(M^l)_{ij} > \varepsilon$.

THEOREM 7. *An irreducible matrix $M \in \mathcal{D}^{n\times n}$ has a unique eigenvalue denoted λ.*

There might be several eigenvectors of an irreducible matrix with the unique corresponding eigenvalue λ. Expressions of eigenvalue and eigenvectors can be found in references above. Let us finally recall that every irreducible matrix is cyclic in the sense of the following theorem (see *e.g.* (Baccelli et al., 1992, 3.7)).

THEOREM 8. *Let $M \in \mathcal{D}^{n\times n}$ be an irreducible matrix whose eigenvalue is λ. There exist integers N and c such that*

$$\forall m \geq N,\ M^{m+c} = \lambda^c \otimes M^m.$$

The least value of c is called the cyclicity of M.

If entries belong to a dioid satisfying the weak stabilization condition (Def. 4), we have the following extension for possibly reducible matrices.

THEOREM 9 (See (De Schutter, 2000; Gaubert, 1994, prop. 1.2.2)). *Let $\mathcal{D}$ be a commutative dioid satisfying the weak stabilization condition and let $M \in \mathcal{D}^{n\times n}$. For all $i, j \in \{1, \ldots, n\}$, there exist $c \in \mathbb{N} \setminus \{0\}$, $\lambda_0, \ldots, \lambda_{c-1} \in \mathcal{D}$, $N \in \mathbb{N}$ such that*

$$\forall m \geq N,\ \left[M^{m+l+c}\right]_{ij} = \lambda_l^c \left[M^{m+l}\right]_{ij} \quad \textit{for } l = 0, \ldots, c-1\ .$$

2.2. Signals and systems

Let $\mathcal{D}$ be a complete dioid. A signal is here supposed to be a function from $\mathbb{Z}$ into $\mathcal{D}$. Whereas in the conventional system theory the set of signals is endowed with a vector space structure, in our context, the set of signals $\mathcal{D}^{\mathbb{Z}}$ is equipped with a *moduloid* structure. More precisely, we denote $\mathcal{E}$ the set $\mathcal{D}^{\mathbb{Z}}$ endowed with the two following operations:

- an inner operation, denoted '$\oplus$', which plays the role of addition of signals: $\forall u, v \in \mathcal{D}^{\mathbb{Z}}$, $\forall t \in \mathbb{Z}$; $\quad (u \oplus v)(t) = u(t) \oplus v(t)$;
- an external operation, denoted '$\cdot$', which plays the role of product of a signal with a scalar: $\forall a \in \mathcal{D}$, $\forall v \in \mathcal{D}^{\mathbb{Z}}$, $\forall t \in \mathbb{Z}$; $\quad (a \cdot v)(t) = a \otimes v(t)$.

DEFINITION 10. [Linear system] A p-input, q-output system is a mapping $\mathcal{S} : \mathcal{E}^p \to \mathcal{E}^q$, $u \mapsto y$. It is called linear over dioid $(\mathcal{D}, \oplus, \otimes)$ if

$$\begin{array}{lll} & \forall u,\ v \in \mathcal{E}^p\,; & \mathcal{S}(u \oplus v) = \mathcal{S}(u) \oplus \mathcal{S}(v)\,, \\ \textit{and} & \forall u \in \mathcal{E}^p,\ \forall a \in \mathcal{D}\,; & \mathcal{S}(a \cdot u) = a \cdot \mathcal{S}(u)\,. \end{array} \tag{1}$$

Classically, an additional continuity assumption is made for the considered systems. Namely, we require that for any finite, or infinite, collection $(u_i)_{i \in I}$

$$\mathcal{S}(\bigoplus_{i \in I} u_i) = \bigoplus_{i \in I} \mathcal{S}(u_i)\,.$$

EXAMPLE 11. An elementary linear system is the system denoted Δ^s, whose output y is equal to its input u delayed by s ($s \in \mathbb{Z}$) :

$$\forall t \in \mathbb{Z}, \quad y(t) = [\Delta^s(u)]\,(t) = u(t - s)\,.$$

DEFINITION 12. A linear system $\mathcal{S}$ is *causal* if for all inputs u_1 and u_2

$$\forall \tau \in \mathbb{Z}, \quad u_1(t) = u_2(t) \text{ for } t \leq \tau \quad \Rightarrow \quad [\mathcal{S}(u_1)]\,(t) = [\mathcal{S}(u_2)]\,(t) \text{ for } t \leq \tau\,.$$

DEFINITION 13. A linear system $\mathcal{S}$ is *time-invariant* if

$$\forall u \in \mathcal{E},\ \forall s \in \mathbb{Z}\,; \quad \mathcal{S}(\Delta^s(u)) = \Delta^s \mathcal{S}(u)\,,$$

in which Δ^s is the elementary delay system (see ex. 11).

2.3. Representations of linear systems

2.3.1. *Impulse response*

It is well known that any linear system can be represented by a unique mapping called *impulse response*. With the aim of doing so over a dioid $\mathcal{D}$, we define the Dirac function at 0 and its shifted versions as

$$e\ :\ t \to e(t) = \begin{cases} e & \text{if } t = 0, \\ \varepsilon & \text{otherwise;} \end{cases} \quad \text{and} \quad \forall t \in \mathbb{Z}\,, \quad \delta^s(t) = \Delta^s(e(t)) = e(t - s)\,.$$

Hence, it can be checked by direct calculation that

$$\forall u \in \mathcal{E},\ \forall t \in \mathbb{Z}\,; \quad u(t) = \bigoplus_{s \in \mathbb{Z}} u(s) e(t - s) = \bigoplus_{s \in \mathbb{Z}} u(s) \delta^s(t)\,.$$

Starting from this decomposition of signals, the impulse response of a linear system has been defined as in the following theorem.

THEOREM 14 (See (Baccelli et al., 1992, 6.3.1)). *Let $\mathcal{S}$ be a linear system, then there exists a unique mapping $h : \mathbb{Z}^2 \mapsto \mathcal{D}^{q\times p}$, (called impulse response) defined by*

$$h(t,s) = [\mathcal{S}(\delta^s)](t) , \tag{2}$$

such that

$$\forall u \in \mathcal{E}^p,\ \forall t \in \mathbb{Z} ; \quad y(t) = [\mathcal{S}(u)](t) = \bigoplus_{s\in\mathbb{Z}} h(t,s)u(s) .$$

2.3.2. *State space representation*

State space representations of linear systems over dioids have also been studied. In particular, a p-input, q-output system is said to be linear and causal if it can be represented by the following state space model

$$x(t) = A(t-1)x(t-1) \oplus B(t)u(t) \tag{3}$$

$$y(t) = C(t)x(t) \tag{4}$$

in which

- $A(t) \in \mathcal{D}^{n\times n}$, $B(t) \in \mathcal{D}^{n\times p}$ and $C(t) \in \mathcal{D}^{q\times n}$.
- $u(t) \in \mathcal{D}^p$ (respectively $x(t) \in \mathcal{D}^n$, $y(t) \in \mathcal{D}^q$) is called the *input* (respectively *state*, *output*) vector.

The solution to Eq. (3) is

$$t > t_0, \quad x(t) = \Phi(t,t_0)x(t_0) \oplus \bigoplus_{j=t_0+1}^{t} \Phi(t,j)B(j)u(j) ,$$

in which $\Phi(t,t_0)$ is called *transition matrix* by analogy with conventional time-varying linear system theory (Kailath, 1980; Kamen, 1996) and is defined by

$$\Phi(t,t_0) = \begin{cases} \text{not defined} & \text{for } t < t_0, \\ \mathsf{Id}_n & \text{for } t = t_0, \\ A(t-1)A(t-2) \otimes \cdots \otimes A(t_0) & \text{for } t > t_0. \end{cases} \tag{5}$$

By definition, this matrix satisfies the composition property:

$$t \geq j \geq t_0, \quad \Phi(t,t_0) = \Phi(t,j) \otimes \Phi(j,t_0) , \tag{6}$$

and in addition,

$$t > t_0, \quad \Phi(t,t_0) = A(t-1)\Phi(t-1,t_0) . \tag{7}$$

The *input-output relationship* is simply deduced from the output equation (Eq. (4)):

$$\forall t \in \mathbb{Z}, \quad y(t) = \bigoplus_{s\in\mathbb{Z}} h(t,s)u(s) \tag{8}$$

where h, the *impulse response*, is defined by

$$h(t,s) = \begin{cases} C(t)\Phi(t,s)B(s) & ,t \geq s, \\ \varepsilon & ,t < s. \end{cases} \tag{9}$$

3. Linear periodic systems

In this very section the study of linear periodic systems is tackled. At first, linear periodic systems are defined as systems which commute with the delay operator on signals for delays corresponding to their period. Secondly, we exhibit several properties of representations associated with such systems. Towards their impulse responses, a necessary and sufficient condition is given to characterize the periodicity. We also introduce the notion of *monodromy matrix* in state space representation. Spectral properties of the monodromy matrix of an autonomous periodic system notably allow showing that its state couples with a periodic steady state in finite time.

DEFINITION 15. A system $\mathcal{S}$ is called periodic of period T (or shortly T-periodic), if T is the least positive integer such that

$$\forall u \in \mathcal{E}^p, \quad \mathcal{S}(\Delta^T(u)) = \Delta^T\mathcal{S}(u) \,,$$

where Δ is the delay operator on signals (see ex. 11).

From this definition, one can easily deduce that a T-periodic system $\mathcal{S}$ satisfies :

$$\forall u \in \mathcal{E}^p, \forall n \in \mathbb{Z}\,; \quad \mathcal{S}(\Delta^{nT}(u)) = \Delta^{nT}\mathcal{S}(u)\,. \tag{10}$$

Remark 1. Whereas a time-invariant system $\mathcal{S}$ (see def. 13) commutes with operator Δ^s for all values of delay $s \in \mathbb{Z}$, *i.e.*

$$\mathcal{S}\left(\Delta^s(u)\right) = \Delta^s\left(\mathcal{S}(u)\right) \,,$$

a T-periodic system commutes with Δ^s only for s multiple of period T. Moreover, a time-invariant system can be seen as a 1-periodic system.

3.1. Impulse response

The next proposition provides a condition on its impulse response which characterizes the T-periodicity of a linear system.

PROPOSITION 16. *A linear system $\mathcal{S}$ is T-periodic if, and only if, T is the least positive integer such that its impulse response h satisfies*

$$\forall t,s \in \mathbb{Z}, \quad h(t+T, s+T) = h(t,s)\,. \tag{11}$$

Proof. Necessary condition: Suppose that system $\mathcal{S}$ is T-periodic, we get

$$\begin{aligned} h(t+T,s+T) &= \left[\mathcal{S}(\delta^{s+T})\right](t+T) && \text{(by definition of } h, \text{ see Eq. (2))} \\ &= \left[\Delta^{-T}\left(\mathcal{S}(\delta^{s+T})\right)\right](t) \\ &= \left[\mathcal{S}\left(\Delta^{-T}(\delta^{s+T})\right)\right](t) && \text{(}\mathcal{S} \text{ is } T\text{-periodic, see Eq. (10))} \\ &= \left[\mathcal{S}(\delta^{s})\right](t) \\ &= h(t,s)\,. \end{aligned}$$

Sufficient condition: As

$$h(t+T,s+T) = \left[\mathcal{S}(\delta^{s+T})\right](t+T) = \left[\Delta^{-T}\left(\mathcal{S}(\delta^{s+T})\right)\right](t),$$

and

$$h(t,s) = \left[\mathcal{S}(\delta^{s})\right](t) = \left[\mathcal{S}\left(\Delta^{-T}(\delta^{s+T})\right)\right](t),$$

equality (11) implies that $\mathcal{S}$ is T-periodic.

3.2. State space representation

A linear and causal system described by a state space representation given by Eqs. (3)-(4) is said T-periodic if T is the least positive integer such that

$$\forall t \in \mathbb{Z}, \quad A(t+T) = A(t), \quad B(t+T) = B(t), \quad C(t+T) = C(t)\,.$$

LEMMA 17. *Transition matrix Φ defined by (5) is T-periodic, i.e.*

$$t \geq t_0, \quad \Phi(t+T, t_0+T) = \Phi(t,t_0)\,,$$

if, and only if, $A(t)$, $t \in \mathbb{Z}$, is T-periodic.

Proof. Straightforward.

The T-periodicity of Φ also allows writing

$$t \geq t_0, \forall m \in \mathbb{Z}, \quad \Phi(t+mT, t_0+mT) = \Phi(t,t_0)\,. \tag{12}$$

Setting $t = i + nT$ with $T > i \geq t_0$ and $n \in \mathbb{N}$, we get

$$\begin{aligned} &\Phi(i+nT,t_0) \\ &\quad = \Phi(i+nT, t_0+nT)\Phi(t_0+nT, t_0+(n-1)T) \otimes \ldots \otimes \Phi(t_0+T,t_0) && \text{(see Eq. (6))} \\ &\quad = \Phi(i,t_0)\,\underbrace{\Phi(t_0+T,t_0) \otimes \ldots \otimes \Phi(t_0+T,t_0)}_{\text{n times}} && \text{(see Eq. (12))} \\ &\quad = \Phi(i,t_0)[\Phi(t_0+T,t_0)]^n\,. \end{aligned}$$

DEFINITION 18. By analogy with conventional theory (Bittanti, 1996; Bolzern et al., 1986), matrix $M_{t_0} = \Phi(t_0+T,t_0)$ is called *monodromy matrix* at t_0.

For autonomous systems, that is systems for which the input is zero ($u(t) = \varepsilon$, $\forall t \in \mathbb{Z}$ in Eq. (3)), the state vector obeys:

$$x(i+nT) = \Phi(i+nT, t_0)x(t_0) = \Phi(i, t_0)[\Phi(t_0 + T, t_0)]^n x(t_0) = \Phi(i, t_0)M_{t_0}^n x(t_0)\,. \quad (13)$$

In other words, the monodromy matrix describes the evolution of the state over one period. This relation allows showing that an autonomous periodic system couples to a periodic regime in finite time.

PROPOSITION 19. *If the monodromy matrix M_{t_0} is irreducible with eigenvalue λ, then there exist two integers N and c such that for $n \geq N$*

$$x(i + (n + c)T) = \lambda^c x(i + nT)\,. \quad (14)$$

Proof. From Eq. (13), a direct application of Th. 8, for n large enough, leads to

$$\begin{aligned} x(i + (n + c)T) &= \Phi(i, t_0)M_{t_0}^{n+c} x(t_0) \\ &= \Phi(i, t_0)\lambda^c M_{t_0}^n x(t_0) \\ &= \lambda^c \Phi(i, t_0)M_{t_0}^n x(t_0) \\ &= \lambda^c x(i + nT) \end{aligned}$$

in which λ (resp. c) is the eigenvalue (resp. the cyclicity) of M_{t_0}.

Remark 2. If the dioid, in which the signals take their values, satisfies the weak stabilization condition (see Def. 4), the existence of a steady state can be shown even if the monodromy matrix is reducible. Under this assumption, theorem 9 indeed applies and similarly a steady state can then be identified.

PROPOSITION 20. *The spectrum of $M_{t_0} = \Phi(t_0 + T, t_0)$ is independent of t_0. Furthermore, if $x(t_0)$ is an eigenvector of M_{t_0} with corresponding eigenvalue λ, then $x(t) = \Phi(t, t_0)x(t_0)$ is an eigenvector of $M_t = \Phi(t+T, t)$ with corresponding eigenvalue λ.*

Proof.

- For any pair (t_0, t_1) with $t_0 + T \geq t_1 \geq t_0$, the monodromy matrices at t_0 and at t_1 can respectively be written
  - $\Phi(t_0 + T, t_0) = \Phi(t_0 + T, t_1)\Phi(t_1, t_0)$ (see Eq. (6)) .
  - $\Phi(t_1 + T, t_1) = \Phi(t_1 + T, t_0 + T)\Phi(t_0 + T, t_1)$ (see Eq. (6))
    $= \Phi(t_1, t_0)\Phi(t_0 + T, t_1)$ (Φ is periodic) .

  In other words, by setting $F = \Phi(t_1, t_0)$ and $G = \Phi(t_0 + T, t_1)$, these monodromy matrices can be written: $\Phi(t_0 + T, t_0) = GF$ and $\Phi(t_1 + T, t_1) = FG$.
  If λ is a (nonzero) eigenvalue of $\Phi(t_1 + T, t_1)$, *i.e.*, $FGx = \lambda x$, $x \neq \varepsilon$, then $GFGx = G\lambda x = \lambda Gx$, or $GFy = \lambda y$ with $y = Gx$.
  Since $\lambda \neq \varepsilon$ and $x \neq \varepsilon$, $y = Gx \neq \varepsilon$; so that λ is an eigenvalue of $\Phi(t_0+T, t_0) = GF$ as well.
  By reversing the role of $\Phi(t_0 + T, t_0)$ and $\Phi(t_1 + T, t_1)$ in the above argument,

it conversely follows that all the (nonzero) eigenvalues of $\Phi(t_0+T,t_0)$ are also eigenvalues of $\Phi(t_1+T,t_1)$.

- Let us assume that $x(t_0)$ is an eigenvector of M_{t_0} with corresponding eigenvalue λ. We have for $t \geq t_0$

$$\begin{aligned} M_t x(t) = \Phi(t+T,t)x(t) &= \Phi(t+T,t)\Phi(t,t_0)x(t_0) \\ &= \Phi(t+T,t_0)x(t_0) \\ &= \Phi(t+T,t_0+T)\Phi(t_0+T,t_0)x(t_0) \quad \text{(see Eq.(6))} \\ &= \Phi(t,t_0)\lambda x(t_0) \\ &= \lambda x(t) \end{aligned}$$

which shows that $x(t)$ is an eigenvector of M_t with eigenvalue λ.

Remark 3. In (Lahaye, 2000), potential applications to manufacturing systems have been studied. In particular, an automobile production line and an electronic cards production line have been modeled by a state representation (3-4) in dioid $\overline{\mathbb{R}}_{\max}$ in which

– variables, corresponding to entries of vectors $u(t)$, $x(t)$ and $y(t)$, denote input release dates of successive products (electronic cards or automobiles) at distinct stages of assembly lines,

– coefficients of matrices $A(t)$, $B(t)$ and $C(t)$ correspond to the processing times of successive products at different levels of production lines. These parameters depend upon the types of products processed.

In these manufacturing systems, production is usually *repetitive*, that is, the types of products successively released on the production line are ordered in a periodic manner. Then these systems can be studied as periodic systems in $\overline{\mathbb{R}}_{\max}$ since matrices $A(t)$, $B(t)$, $C(t)$ admit periodic variations.
When they are maximally solicited (release dates of products are considered to be equal to $-\infty$, *i.e.*, $u(t)=\varepsilon$, $\forall t \in \mathbb{Z}$), proposition 19 allows claiming that they couple to a periodic regime. More precisely, their state vector satisfies Eq. (14) in $\overline{\mathbb{R}}_{\max}$ for n large enough; in conventional algebra, this leads to:

$$x(i+(n+c)T) = c \times \lambda + x(i+nT)\ .$$

The value $(c\times\lambda)/(c\times T) = \lambda/T$ gives to the cycle time of the system, that is, the mean production time of an electronic card or a vehicle. The product $c \times T$ is equal to the length of the periodic pattern. Finally, proposition 20 allows claiming that the cycle time is identical for any circular permutation of the periodic sequence of production. Such applications in performance evaluation of assembly lines may be used to ease scheduling and/or design decisions since they enable the evaluation of the relevance of different solutions (scheduling and/or re-engineering of production lines).

4. State-space realization for linear periodic systems

Consider that we only know the impulse response of a periodic system. The *state space realization problem* tackled in this section aims at constructing a periodic state space model of this system. Actually, we only address the starting point of such an approach. In fact, being given the impulse response of a periodic system (characterized by Eq. (11)), we exhibit a necessary and sufficient condition for the existence of a periodic realization.
As for conventional periodic systems (Bru et al., 1997), the realization problem is studied by means of a time-invariant reformulation of their periodic state space representation. This so-called *cyclic reformulation* has been defined by analogy with conventional theory (Misra, 1996) and is introduced in a first place.

4.1. Cyclic time-invariant reformulation of a periodic state space representation

First of all, we point out some properties of the $kT \times kT$ matrix denoted P_k and defined by

$$P_k = \begin{pmatrix} \varepsilon & \varepsilon & \dots & \varepsilon & \mathsf{Id}_k \\ \mathsf{Id}_k & \varepsilon & \ddots & \varepsilon & \varepsilon \\ \varepsilon & \mathsf{Id}_k & \ddots & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \ddots & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \ddots & \mathsf{Id}_k & \varepsilon \end{pmatrix} . \tag{15}$$

Matrix P_k satisfies:

$$P_k^T = \mathsf{Id}_{kT} \ , \ \text{ and consequently } \ P_k^{T+j} = P_k^j \ , \ j \in \mathbb{N}. \tag{16}$$

LEMMA 21. *Let $F(t)$, $t \in \mathbb{Z}$, be a T-periodic $l \times m$ matrix, and $\hat{F}(t)$ the $lT \times mT$ block-diagonal matrix defined by*

$$\hat{F}(t) = \mathsf{diag}\big(F(t), F(t+1), \dots, F(t+T-1)\big) \ .$$

We have

$$\hat{F}(i) \otimes P_m^j = P_l^j \otimes \hat{F}(i+j) \ . \tag{17}$$

Proof. We shall prove (17) by induction on index j. Using the T-periodicity of matrix F, one checks by direct calculation that (17) is satisfied for $j = 1$, *i.e.*:$\hat{F}(i) \otimes P_m = P_l \otimes \hat{F}(i+1)$. Suppose that (17) is satisfied for $j = n$, *i.e.*: $\hat{F}(i) \otimes P_m^n = P_l^n \otimes \hat{F}(i+n)$. We show that (17) is also satisfied for $j = n+1$:

$$\begin{aligned} \hat{F}(i) \otimes P_m^{n+1} &= \hat{F}(i) \otimes P_m^n \otimes P_m \\ &= P_l^n \otimes \hat{F}(i+n) \otimes P_m && \text{(by assumption)} \\ &= P_l^n \otimes P_l \otimes \hat{F}(i+n+1) && \text{(since (17) is true for } j=1) \\ &= P_l^{n+1} \otimes \hat{F}(i+(n+1)) \ . \end{aligned}$$

PROPOSITION 22. *Let $\big(A(t),\ B(t),\ C(t)\big)$, $t \in \mathbb{Z}$, be a realization of a linear T-periodic system and t_0 an initial time instant. Signals $\tilde{x}$, $\tilde{u}$ and $\tilde{y}$ defined by*

$$\tilde{x}(t) = P_n^{t-t_0} \otimes \Big(x(t)^\top \ \ x(t+1)^\top \ \ \ldots \ \ x(t+T-1)^\top\Big)^\top ,$$

$$\tilde{u}(t) = P_p^{t-t_0} \otimes \Big(u(t)^\top \ \ u(t+1)^\top \ \ \ldots \ \ u(t+T-1)^\top\Big)^\top ,$$

$$\tilde{y}(t) = P_q^{t-t_0} \otimes \Big(y(t)^\top \ \ y(t+1)^\top \ \ \ldots \ \ y(t+T-1)^\top\Big)^\top$$

satisfy the following time-invariant state space representation

$$\tilde{x}(t) \ = \ \tilde{A}\tilde{x}(t-1) \oplus \tilde{B}\tilde{u}(t) \tag{18}$$
$$\tilde{y}(t) \ = \ \tilde{C}\tilde{x}(t) \tag{19}$$

in which

$$\tilde{A} = \begin{pmatrix} \varepsilon & \ldots & \varepsilon & A(t_0+T-1) \\ A(t_0) & \ddots & \varepsilon & \varepsilon \\ \varepsilon & \ddots & \varepsilon & \varepsilon \\ \varepsilon & \ddots & A(t_0+T-2) & \varepsilon \end{pmatrix}, \tag{20}$$

$$\tilde{B} = \mathsf{diag}\big(B(t_0),\ldots,B(t_0+T-1)\big),\ \tilde{C} = \mathsf{diag}\big(C(t_0),\ldots,C(t_0+T-1)\big)\ . \tag{21}$$

Proof. By stacking equations of the state space model of a periodic system for instants $t, t+1, \ldots, t+T-1$, we get an equivalent periodic state space representation defined by

$$\hat{x}(t) \ = \ \hat{A}(t-1)\hat{x}(t-1) \oplus \hat{B}(t)\hat{u}(t) \tag{22}$$
$$\hat{y}(t) \ = \ \hat{C}(t)\hat{x}(t) \tag{23}$$

with
$\hat{x}(t) = \Big(x(t)^\top \ \ x(t+1)^\top \ \ \ldots \ \ x(t+T-1)^\top\Big)^\top, \hat{A}(t) = \mathsf{diag}\big(A(t),\ldots,A(t+T-1)\big)$,
$\hat{u}(t) = \Big(u(t)^\top \ \ u(t+1)^\top \ \ \ldots \ \ u(t+T-1)^\top\Big)^\top, \hat{B}(t) = \mathsf{diag}\big(B(t),\ldots,B(t+T-1)\big)$,
$\hat{y}(t) = \Big(y(t)^\top \ \ y(t+1)^\top \ \ \ldots \ \ y(t+T-1)^\top\Big)^\top, \hat{C}(t) = \mathsf{diag}\big(C(t),\ldots,C(t+T-1)\big)$.
Let us set

$$\begin{cases} \tilde{x}(t) \ = \ P_n^{t-t_0} \otimes \hat{x}(t) \\ \tilde{u}(t) \ = \ P_p^{t-t_0} \otimes \hat{u}(t) \\ \tilde{y}(t) \ = \ P_q^{t-t_0} \otimes \hat{y}(t) \end{cases}$$

or equivalently with $t = i + nT$, $t_0 + T > i \geq t_0$, and thanks to Eqs. (16)

$$\begin{cases} \tilde{x}(i+nT) \ = \ P_n^{i+nT-t_0} \otimes \hat{x}(i+nT) \ = \ P_n^{i-t_0} \otimes \hat{x}(i+nT) \\ \tilde{u}(i+nT) \ = \ P_p^{i+nT-t_0} \otimes \hat{u}(i+nT) \ = \ P_p^{i-t_0} \otimes \hat{u}(i+nT) \\ \tilde{y}(i+nT) \ = \ P_q^{i+nT-t_0} \otimes \hat{y}(i+nT) \ = \ P_q^{i-t_0} \otimes \hat{y}(i+nT) \end{cases} .$$

Multiplying each equation respectively by $P_n^{T-i+t_0}$, $P_p^{T-i+t_0}$ and $P_q^{T-i+t_0}$, as well as using Eqs. (16), we get

$$\begin{cases} \hat{x}(i+nT) &= P_n^{T-i+t_0} \otimes \tilde{x}(i+nT) \\ \hat{u}(i+nT) &= P_p^{T-i+t_0} \otimes \tilde{u}(i+nT) \\ \hat{y}(i+nT) &= P_q^{T-i+t_0} \otimes \tilde{y}(i+nT) \end{cases}$$

Since $\hat{A}(\cdot)$, $\hat{B}(\cdot)$, $\hat{C}(\cdot)$ are periodic, the state space model defined by (22)-(23) can then be written

$$\begin{array}{l} P_n^{T-i+t_0}\tilde{x}(i+nT) = \hat{A}(i-1)P_n^{T-i+t_0+1}\,\tilde{x}(i+nT-1) \oplus \hat{B}(i)P_p^{T-i+t_0}\tilde{u}(i+nT) \\ P_q^{T-i+t_0}\tilde{y}(i+nT) = \hat{C}(i)P_n^{T-i+t_0}\tilde{x}(i+nT)\,, \end{array}$$

or, multiplying the state equation by $P_n^{i-t_0}$ and the output equation by $P_q^{i-t_0}$,

$$\begin{array}{l} \tilde{x}(i+nT) = P_n^{i-t_0}\hat{A}(i-1)P_n^{T-i+t_0+1}\tilde{x}(i+nT-1) \oplus P_n^{i-t_0}\hat{B}(i)P_p^{T-i+t_0}\tilde{u}(i+nT) \\ \tilde{y}(i+nT) = P_q^{i-t_0}\hat{C}(i)P_n^{T-i+t_0}\tilde{x}(i+nT)\,. \end{array}$$

By using (17), we obtain

$$\begin{aligned} P_n^{i-t_0}\hat{A}(i-1)P_n^{T-i+t_0+1} &= P_n^{i-t_0}P_n^{T-i+t_0+1}\hat{A}(i-1+T-i+t_0+1) \\ &= P_n^{T+1}\hat{A}(T+t_0) \\ &= P_n\hat{A}(t_0) \quad \text{(from Eqs. (16) and since } \hat{A} \text{ is periodic)} \\ &= \tilde{A}\,. \end{aligned}$$

Similar arguments lead to equalities

$$P_n^{i-t_0}\hat{B}(i)P_p^{T-i+t_0} = \tilde{B}\,, \qquad P_q^{i-t_0}\hat{C}(i)P_n^{T-i+t_0} = \tilde{C}\,.$$

Which finally shows that $\tilde{x}$, $\tilde{u}$ and $\tilde{y}$ satisfy Eqs. (18)-(19).

Although not explicitly specified, time-invariant realization $(\tilde{A}, \tilde{B}, \tilde{C})$ depends on the initial time instant t_0. Since its realization $(A(t), B(t), C(t))$, $t \in \mathbb{Z}$, is periodic, it is obvious that a T-periodic system has only T different time-invariant realizations $(\tilde{A}, \tilde{B}, \tilde{C})$.

4.2. State space realization

The state space realization has previously been studied for linear time-invariant systems over dioid $\overline{\mathbb{R}}_{\max}$ (Olsder, 1986; De Schutter and De Moor, 1995). In particular, being given the impulse response of a $(\max, +)$ linear time-invariant system, a necessary and sufficient condition for the existence of a state realization has been stated. Making use of the cyclic time-invariant reformulation, the following proposition extends this condition to periodic systems.

PROPOSITION 23. *Let $\mathcal{D}$ be a commutative dioid satisfying the weak stabilization condition. A mapping $h : \mathbb{Z}^2 \to \mathcal{D}^{q\times p}$, satisfying Eq. (11) is the impulse response of a system represented by a T-periodic realization $(A(t), B(t), C(t))$, $t \in \mathbb{Z}$, if, and only if,*

$\forall t_0 \in \mathbb{Z}$, $\forall i \in \{1, \ldots, q\}$, $\forall j \in \{1, \ldots, p\}$,
$\exists c \in \mathbb{N} \setminus \{0\}$, $\exists \lambda_0, \ldots, \lambda_{c-1} \in \mathcal{D}$, $\exists N \in \mathbb{N}$
such that $\forall m \geq N$, $\forall l \in \{0, \ldots, c-1\}$, $\forall s \in \{0, \ldots, T-1\}$

$$\left[h(t_0 + s + (m + l + c)T, t_0)\right]_{ij} = \lambda_l^c \left[h(t_0 + s + (m + l)T, t_0)\right]_{ij} . \tag{24}$$

Proof. The proof is quite lengthy and technical. For the sake of clarity, we only give here a sketch of proof (see appendix for a complete account).
Necessary condition comes down to checking that the impulse response of any T-periodic realization $(A(t), B(t), C(t))$, $t \in \mathbb{Z}$ satisfies Eq. (24). With this end in view, theorem 9 must be applied to the time-invariant cyclic reformulation defined in proposition 22 for any T-periodic realization.
Being given a mapping $h : \mathbb{Z}^2 \to \mathcal{D}^{q\times p}$ satisfying both Eq. (11) and the condition given by Eq. (24), sufficient condition consists in finding a periodic state realization of h. In particular, in the *Single Input Single Output* case, it can be shown that the following realization is suitable. For all $t_0 \in \mathbb{Z}$, we define a realization $(A(t_0), B(t_0), C(t_0))$, in which $A(t_0)$ is a $(c^2 + N)T \times (c^2 + N)T$ matrix,

$$A(t_0) = \begin{pmatrix} A_{\lambda_{c-1}} & \varepsilon & \ldots & & \varepsilon \\ \varepsilon & A_{\lambda_{c-2}} & \ddots & & \vdots \\ \vdots & & \ddots & & \vdots \\ \varepsilon & \ldots & \varepsilon & A_{\lambda_0} & \varepsilon \\ Q & \ldots & \ldots & Q & A_N \end{pmatrix} , \tag{25}$$

in which

- Q is a $NT \times cT$ matrix whose entries are equal to ε except for $Q_{1,cT} = e$.
- A_N and each A_{λ_l}, $l = 0, 1, \ldots, c-1$ are respectively a $NT \times NT$ sub-diagonal matrix, and a $cT \times cT$ matrix with the following Frobenius normal form

$$A_N = \begin{pmatrix} \varepsilon & \ldots & & \ldots & \varepsilon \\ e & \ddots & & & \vdots \\ \varepsilon & e & & & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \varepsilon & \ldots & \varepsilon & e & \varepsilon \end{pmatrix} , A_{\lambda_l} = \begin{pmatrix} \varepsilon & \ldots & \ldots & \varepsilon & \lambda_l^c \\ e & \ddots & & & \varepsilon \\ \varepsilon & e & & & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \varepsilon & \ldots & \varepsilon & e & \varepsilon \end{pmatrix} .$$

$C(t_0)$ is a $(c^2 + N)T$ row vector defined as follows

$$C(t_0) = (\varepsilon \;\ldots\; \varepsilon \;\; e\,) . \tag{26}$$

The column vector $B(t_0)$ is defined by

$$B(t_0) = \left(B_{\lambda_{c-1}}^{\top} \; B_{\lambda_{c-2}}^{\top} \; \ldots \; B_{\lambda_0}^{\top} \; B_N^{\top} \right)^{\top} , \tag{27}$$

in which

- B_N is a NT column vector

$$B_N = \left(h(t_0 + NT - 1, t_0) \;\; h(t_0 + NT - 2, t_0) \;\; \ldots \;\; h(t_0, t_0) \right)^{\top} ,$$

- each block B_{λ_l}, $l = 0, 1, \ldots, c-1$, is a $cT \times 1$ vector
$B_{\lambda_l} =$
$$\Big(\underbrace{\varepsilon \; \ldots \; \varepsilon}_{lT \text{ times}} \;\; h(t_0 + (N+l)T + T - 1, t_0) \;\; \ldots \;\; h(t_0 + (N+l)T, t_0) \;\; \underbrace{\varepsilon \; \ldots \; \varepsilon}_{(c-l-1)T \text{ times}} \Big)^{\top} .$$

Let us note that the 'sufficient condition' part of the preceding proof provides a method for constructing a periodic realization $(A(t), B(t), C(t))$, $t \in \mathbb{Z}$, of an impulse response satisfying Eq. (24). This construction of state matrices can be seen as a first step towards a parametric identification of periodic systems. In addition, it is usually required that dimensions of matrices $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ must be as small as possible. This problem is not tackled in this paper. Let us note that this problem has been extensively studied for $(\max, +)$ linear time invariant systems (Olsder, 1986; De Schutter and De Moor, 1995; Olsder and Schutter, 1999), and it has been shown to be NP-hard (Blondel and Portier, 1999).

5. Conclusions and further research

We have studied linear periodic systems for which the underlying algebra is a dioid. A formal definition as well as a characterization of their impulse response have been proposed. Several notions and results from conventional periodic system theory are transposed to the algebraic context of dioids: the *monodromy matrix* whose spectral properties are used to study steady states of autonomous systems, the *cyclic time invariant reformulation* of a periodic state space reformulation which is exploited to study state space realization of periodic systems.
We are inclined to think that further results established for conventional periodic systems could be transposed or adapted for 'our' periodic systems. In particular, the model matching problem, which has been solved for conventional periodic systems in (Colaneri and Kucera, 1997) by means of the cyclic time-invariant reformulation, seems to be a promising direction of investigation. Such a study would constitute an extension of results proposed in (Cottenceau et al., 1999; Cottenceau et al., 2001) for linear time-invariant systems over dioids.

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Appendix

Proof of Prop. 23. Necessary condition: Let $(A(t), B(t), C(t))$, $t \in \mathbb{Z}$, be a T-periodic realization of impulse response h (*i.e.* $h(t, s)$ is defined by (9) for all $t, s \in \mathbb{Z}$). By using proposition 22, for each $t_0 \in \mathbb{Z}$, this realization admits a time-invariant cyclic reformulation $(\tilde{A}, \tilde{B}, \tilde{C})$ with corresponding impulse response

$$\tilde{h}(t) = \begin{cases} \tilde{C}\tilde{A}^t\tilde{B} & , t \geq t_0, \\ \varepsilon & , t < t_0. \end{cases}$$

Let us observe that $\tilde{A}^{mT} = \mathsf{diag}(A_1, A_2, \ldots, A_T)$, for $m \geq 1$, where the diagonal blocks can be written

$A_1 = A(t_0 + T - 1) \otimes \ldots \otimes A(t_0) \otimes M_{t_0}^{m-1}$

$A_2 = A(t_0) \otimes M_{t_0}^{m-1} \otimes A(t_0 + T - 1) \otimes \ldots \otimes A(t_0 + 1)$

$A_3 = A(t_0 + 1)A(t_0) \otimes M_{t_0}^{m-1} \otimes A(t_0 + T - 1) \otimes \ldots \otimes A(t_0 + 2)$

$\vdots$

$A_T = A(t_0 + T - 2) \otimes \ldots \otimes A(t_0) \otimes M_{t_0}^{m-1} \otimes A(t_0 + T - 1)$

with $M_{t_0} = A(t_0 + T - 1) \otimes \ldots \otimes A(t_0)$. More generally, any non zero block (*i.e.* whose entries are all different from ε) of matrix $\tilde{A}^{s+mT}$ with $0 < s \leq T$ can be written in a factorized form as

$$F \otimes M_{t_0}^{m-1} \otimes G$$

in which F, G are $n \times n$ matrices with entries in $\mathcal{D}$.

By theorem 9, for all $i, j \in \{1, \ldots, n\}$, there exist $c \in \mathbb{N} \setminus \{0\}$, $\lambda_0, \ldots, \lambda_{c-1} \in \mathcal{D}$, $N \in \mathbb{N}$ such that $\forall m \geq N$, $\forall l \in \{0, \ldots, c-1\}$

$$\left[M_{t_0}^{m+l+c}\right]_{ij} = \lambda_l^c \left[M_{t_0}^{m+l}\right]_{ij}.$$

From the preceding observation, we hence have a more general formulation for matrix $\tilde{A}^{s+mT}$:

$\forall t_0 \in \mathbb{Z}$, $\forall i, j \in \{1, \ldots, nT\}$, $\exists c \in \mathbb{N} \setminus \{0\}$, $\exists \lambda_0, \ldots, \lambda_{c-1} \in \mathcal{D}$, $\exists N \in \mathbb{N}$ such that $\forall m \geq N$, $\forall l \in \{0, \ldots, c-1\}$, $\forall s \in \{0, \ldots, T-1\}$

$$\left[\tilde{A}^{s+(m+l+c)T}\right]_{ij} = \lambda_l^c \left[\tilde{A}^{s+(m+l)T}\right]_{ij},$$

which implies $\forall i \in \{1, \ldots, qT\}$, $\forall j \in \{1, \ldots, pT\}$,

$$\left[\tilde{h}(s + (m + l + c)T)\right]_{ij} = \lambda_l^c \left[\tilde{h}(s + (m + l)T)\right]_{ij}. \tag{28}$$

Let us now observe from Eqs. (20)-(21) that

$$t \geq t_0, \quad \tilde{h}(t) = P_q^t \otimes \mathsf{diag}\big(h(t_0 + t, t_0), h(t_0 + t + 1, t_0 + 1), \ldots, h(t_0 + t + T - 1, t_0 + T - 1)\big).$$

From Eq. (28), we then deduce that

$\forall t_0 \in \mathbb{Z}$, $\forall i \in \{1, \ldots, q\}$, $\forall j \in \{1, \ldots, p\}$,

$\exists c \in \mathbb{N} \setminus \{0\}$, $\exists \lambda_0, \dots, \lambda_{c-1} \in \mathcal{D}$, $\exists N \in \mathbb{N}$
such that $\forall m \geq N$, $\forall l \in \{0, \dots, c-1\}$, $\forall s \in \{0, \dots, T-1\}$

$$\left[P_q^{s+(m+l+c)T} \mathsf{diag}\big(h(t_0+s+(m+l+c)T, t_0), \dots, h(t_0+s+(m+l+c)T+T-1, t_0+T-1)\big)\right]_{ij}$$

$$= \lambda_l^c \left[P_q^{s+(m+l)T} \mathsf{diag}\big(h(t_0+s+(m+l)T, t_0), \dots, h(t_0+s+(m+l)T+T-1, t_0+T-1)\big)\right]_{ij}$$

which, thanks to Eq. (16) applied to matrix P_q and by identification, leads to the condition given by Eq. (24).

Sufficient condition: In the *Single Input Single Output* case, we must check that matrices $A(t_0), B(t_0), C(t_0)$ defined by Eqs. (25-27), constitute a state realization of h.

Although denoted $A(t_0)$ and $C(t_0)$, these matrices, by construction, do not depend on t_0. From Eq. (11) satisfied by h, we obtain that matrix $B(t_0)$ is T-periodic: $B(t_0 + T) = B(t_0)$ for all $t_0 \in \mathbb{Z}$. By direct calculation, we have for all $t_0 \in \mathbb{Z}$

– for $m \leq N$, $s = 0, \dots, T-1$, $l = 0, \dots, c-1$

$$\begin{aligned} & C(t_0+s+(m+l)T)\Phi(t_0+s+(m+l)T, t_0)B(t_0) \\ = \ & C(t_0+s+(m+l)T)A(t_0+s+(m+l)T-1) \otimes \dots \otimes A(t_0)C(t_0) \\ = \ & C(t_0)A(t_0)^{s+(m+l)T}B(t_0) \qquad \text{(since } A(t_0), C(t_0) \text{ do not depend on } t_0\text{)} \\ = \ & h(t_0+s+(m+l)T, t_0) \end{aligned}$$

– for $r \in \mathbb{N}^+$, $s = 0, \dots, T-1$, $l = 0, \dots, c-1$

$$\begin{aligned} & C(t_0+s+(N+cr+l)T)\Phi(t_0+s+(N+cr+l)T, t_0)B(t_0) \\ = \ & C(t_0)A(t_0)^{s+(N+cr+l)T}B(t_0) \\ = \ & (\lambda_l^c)^r h(t_0+s+(N+l)T, t_0) \\ = \ & h(t_0+s+(N+cr+l)T, t_0) \qquad \text{(thanks to condition (24))} \end{aligned}$$

We can consequently conclude that $(A(t_0), B(t_0), C(t_0))$, $t_0 \in \mathbb{Z}$, is a periodic realization of the impulse response h.

For the *Multiple Input Multiple Output* case, consider arbitrary indices $i \in \{1, \dots, q\}$, $j \in \{1, \dots, p\}$, and the mapping $h_{ij}(\cdot, \cdot) = [h(\cdot, \cdot)]_{ij}$. Since h is assumed to satisfy condition (24), obviously mapping h_{ij} also satisfies Eq. (24). From the first part of this proof, h_{ij} admits a periodic realization denoted $(A_{ij}(t_0), B_{ij}(t_0), C_{ij}(t_0))$, $t_0 \in \mathbb{Z}$. If we repeat this reasoning for all pairs of indices (i, j) with $i \in \{1, \dots, q\}$ and $j \in \{1, \dots, p\}$, and if we define block matrices such that

$$A(t_0) = \mathsf{diag}(A_{11}(t_0), A_{12}(t_0), \dots, A_{1p}(t_0), A_{21}(t_0), \dots, A_{qp}(t_0)) \,,$$

$$B(t_0) = \begin{pmatrix} B_{11}(t_0) & \varepsilon & \dots & \varepsilon \\ \varepsilon & B_{12}(t_0) & \dots & \varepsilon \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon & \varepsilon & \dots & B_{1p}(t_0) \\ B_{21}(t_0) & \varepsilon & \dots & \varepsilon \\ \varepsilon & B_{22}(t_0) & \dots & \varepsilon \\ \vdots & \vdots & & \vdots \\ \varepsilon & \varepsilon & \dots & B_{qp}(t_0) \end{pmatrix}, \quad C(t_0) = \begin{pmatrix} C_{11}(t_0) & \dots & C_{1p}(t_0) & \varepsilon & \dots & \varepsilon \\ \varepsilon & \dots & \varepsilon & C_{21}(t_0) & \dots & \varepsilon \\ \vdots & & \vdots & \vdots & & \vdots \\ \varepsilon & \dots & \varepsilon & \varepsilon & \dots & C_{qp}(t_0) \end{pmatrix}.$$

then $(A(t_0), B(t_0), C(t_0))$, $t_0 \in \mathbb{Z}$, is a periodic realization of h.