UPPER BOUNDS FOR THE TRAVEL TIME ON TRAFFIC SYSTEMS

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2. Review in . . .
3. The road . . .
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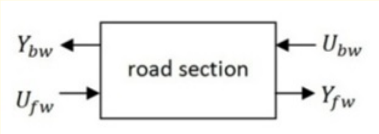
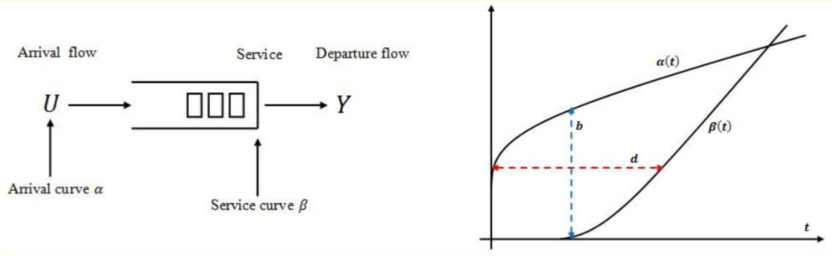
1. Introduction to the approach

- Travel time is one of the key measures of performance and comfort on road networks.
- It is stochastic, highly variable.
- We need good indicators and measures of its variations.
- We derive here upper bounds on the travel time through an itinerary.
- We use an algebraic formulation of the cell-transmission model (Daganzo 1994) on a raod section.





- The dynamics is written linearly in mlin-plus algebra.
- The impulse response of the linear system is interpreted as a service curve in the network calculus theory.
- We then derive an upper bound for the travel time.

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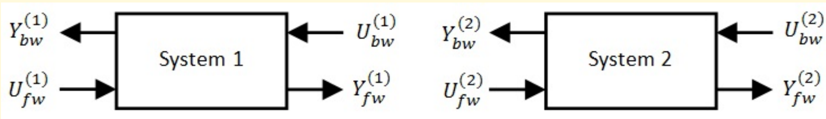

- To extend this drivation , we base on a system theory approach.
- We define elementary traffic systems, and algebraic operators to concatenate them in order to get a large system.
- The concatenation consists in giving the service curve of the large system in function of those of the elementary systems.

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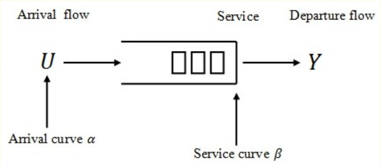
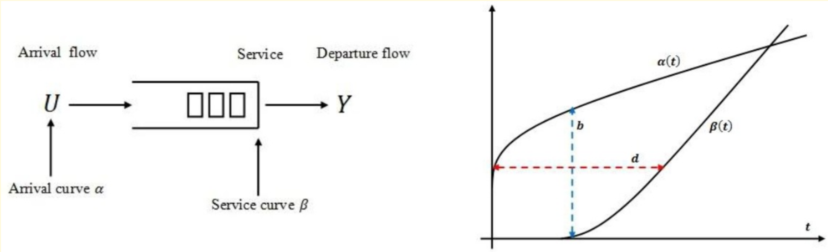
2. Review in deterministic network calculus



- *Arrival curve*. α is an arrival curve if
  $U \leq \alpha * U$ ie $U(t) - U(s) \leq \alpha(t-s), \forall 0 \leq s \leq t$.
- *Service curve*. β is a service curve if
  $Y \geq \beta * U$ ie $Y(t) \geq \min_{0 \leq s \leq t}\{U(s) + \beta(t-s)\}, \forall t \geq 0$.

Two indicators of the service performance:

- The backlog of data in the server $B(t) := U(t) - Y(t)$.
- The virtual delay $d(t) := \inf\{h \geq 0, Y(t+h) \geq U(t)\}$.



Three bounds are obtained:

- The virtual delay is bounded:

$$d(t) \leq \sup_{t \geq 0} \{\inf\{h \geq 0, \beta(t+h) \geq \alpha(t)\}\}, \forall t \geq 0.$$

- The backlog is bounded:

$$B(t) \leq \sup_{s \geq 0} \{\alpha(s) - \beta(s)\}.$$

- The outflow is upper bounded by the arrival curve $\alpha \oslash \beta$:

$$(\alpha \oslash \beta)(t) := \sup_{s \geq 0} \{\alpha(t+s) - \beta(s)\}.$$

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2.1. Matrix arrival.

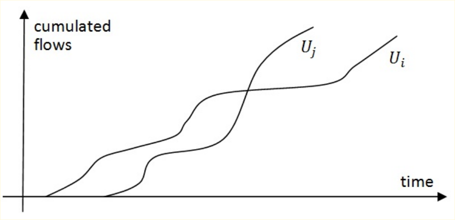
Definition. For a given n-vector U of cumulated arrival flows $U_i, i = 1, 2, \cdots, n$, a $n \times n$ matrix α is said to be a T-arrival matrix for U if

$$\forall i, j, \quad U_i \leq \delta^{-T_{ij}} \alpha_{ij} \otimes U_j,$$

That is

$$\forall i, j, \quad \forall s, t \in \mathbb{N}, \quad U_i(t) - U_j(s) \leq \alpha_{ij}(T_{ij} + t - s).$$

- It is possible to have $U_i(t) - U_j(s) > 0$ even for $s > t$.



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A simple way to obtain such T-arrival matrices, is, first to determine the matrix T (of non negative entries). For a given couple (i, j), T_{ij} is determined as follows.

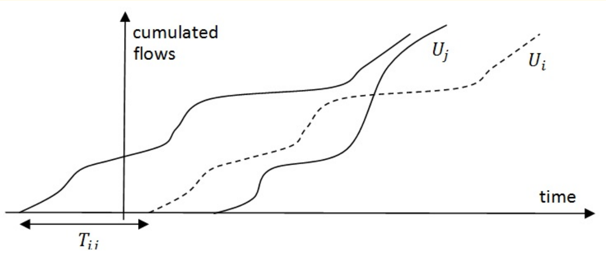
$$T_{ij} = \text{Min}\{\tau \geq 0, U_i(t+\tau) - U_j(t) \leq 0, \forall t \geq 0\}.$$

Then, from Definition 3, α_{ij} satisfy

$$\alpha_{ij} \geq \delta^{T_{ij}} U_i \oslash U_j$$

It is easy to check that for $i = j$, we have $T_{ii} = 0$, and then α_{ii} is a one-dimensional arrival curve for U_i.

Let us notice that Definition 3 is different form Definition 4.2.1 given in [2]. Definition 3 is illustrated in the numerical example of the last section.



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2.2. Service Matrix and virtual delay.

Definition. (service matrix) For a given server with input vector U and output vector Y, a $n \times n$ matrix β is said to be a service matrix for the server, if $Y \geq \beta \times U$.

Definition (virtual delay). For a given server with input vector U and output vector Y, the virtual delay of the last quantity arrived at time t from the ith input to depart from the jth output, denoted $d_i(t)$ is defined:

$$d_i(t) = \inf\{d \geq 0, Y_i(t+d) \geq U_i(t)\}.$$

Theorem 1. For a given server with input vector U and output vector Y, if α is a T-arrival matrix for U, and β is a service matrix for the server, then

$$\forall i = 1,2,\cdots,n, \forall t \in \mathbb{N}, d_i(t) \leq \mathrm{Inf}\{d \geq 0, \alpha_{ij}(T_{ij}+s) \leq \beta_{ij}(s+d), \quad -T_{ij} \leq s \leq t, j = 1,2,\cdots,n\}.$$

and then the virtual delays $d_i, i = 1,\cdots,n$ are bounded as follows.

$$\forall i = 1,2,\cdots,n, \forall t \in \mathbb{N}, \quad d_i(t) \leq \max_{1 \leq j \leq n}\{\sup_{s \geq -T_{ij}}\{inf\{d \geq 0, \alpha_{ij}(T_{ij}+s) \leq \beta_{ij}(s+d)\}\}\}.$$

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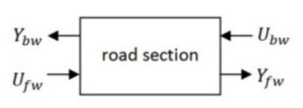
3. The road section model



- U_{fw}: cumulated forward inflow to the section.
- Y_{fw}: cumulated forward outflow from the section.
- U_{bw}: cumulated backward supply from section $i+1$.
- Y_{bw}: cumulated backward supply from section i.

- $Z_{fw} = (Y_{fw} - n)^+ \quad Z_{bw} = (Y_{bw} - \bar{n})^+$.
- Assumption: $U_{fw}(0) = Y_{fw}(0) = U_{bw}(0) = Y_{bw}(0) = 0$.

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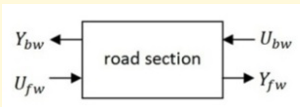
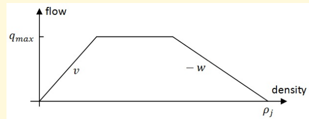
3.1. The traffic dynamics.





- We base on the cell-transmission model (Daganzo 1994).
- with a trapezoidal fundamental traffic diagram.
- We get the dynamics

$$Q(t) = \min\left\{Q\left(t - \frac{\Delta x}{v}\right) + q_{max}\frac{\Delta x}{v}, U_{fw}\left(t - \frac{\Delta x}{v}\right) + n, U_{bw}(t)\right\}$$

$$Y_{fw}(t) = Q(t)$$

$$Y_{bw}(t) = Q\left(t - \frac{\Delta x}{w}\right) + \bar{n}$$

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- By using the min-plus algebra notations, we get

$$
\begin{aligned}
Q &= \gamma^{q_{max}\,\Delta x/v}\delta^{\Delta x/v} Q \oplus \gamma^{n}\delta^{\Delta x/v} U_{fw} \oplus U_{bw} \\
Y_{fw} &= Q \oplus e \\
Y_{bw} &= \gamma^{\bar{n}}\delta^{\Delta x/w} Q \oplus e
\end{aligned}
$$

- We can write

$$
\begin{aligned}
Q &= A * Q \oplus B * U \\
Y &= C * Q \oplus e
\end{aligned}
$$

with $A = \gamma^{q_{max}\,\Delta x/v}\delta^{\Delta x/v}$, $B = \begin{pmatrix} \gamma^{n}\delta^{\Delta x/v} & e \end{pmatrix}$, and $C = \begin{pmatrix} e \\ \gamma^{\bar{n}}\delta^{\Delta x/w} \end{pmatrix}$.

- Then $Y \geq (e \oplus C * A^* * B) * U$.
- Hence $Z \geq H * (e \oplus C * A^* * B) * U$ with

$$
H = \begin{pmatrix} \gamma^{-n} & \varepsilon \\ \varepsilon & \gamma^{-\bar{n}} \end{pmatrix}
$$

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Theorem 2. The matrix $H * (e \oplus C * A^* * B)$ is a service matrix for the road section, seen as a server with two inputs and two outputs.

$$C * A^* * B = \left(\gamma^{q_{max}\,\Delta x/v}\delta^{\Delta x/v}\right)^* * \begin{pmatrix} \gamma^{n}\delta^{\Delta x/v} & e \\ \gamma^{n+\bar{n}}\delta^{\Delta x/v+\Delta x/w} & \gamma^{\bar{n}}\delta^{\Delta x/w} \end{pmatrix}.$$

Corollary 1. A service matrix β for the road section seen as a server, is given as follows.

$$\beta = \begin{pmatrix} q_{max}\left(t - \frac{\Delta x}{v}\right)^+ & q_{max}\left(t - \frac{\Delta x}{v} - \frac{n}{q_{max}}\right)^+ \\ q_{max}\left(t - \frac{\Delta x}{v} - \frac{\Delta x}{v} + \frac{n}{q_{max}}\right)^+ & q_{max}\left(t - \frac{\Delta x}{w}\right)^+ \end{pmatrix}$$

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4. The controlled road section model

- We consider that the section is controlled with a traffic light.
- We denote C: the cycle, G: the green time, R: red time.
- In the dynamics, we replace

$$Q(t) = \min\left\{Q\left(t - \frac{\Delta x}{v}\right) + q_{max}\frac{\Delta x}{v}, U_{fw}\left(t - \frac{\Delta x}{v}\right) + n, U_{bw}(t)\right\}$$

with

$$Q(t) = \min\left\{Q\left(t - \frac{\Delta x}{v}\right) + (G/c)q_{max}\frac{\Delta x}{v}, U_{fw}\left(t - \frac{\Delta x}{v} - R\right) + n, U_{bw}(t)\right\}$$

Theorem 3. The matrix $H * (e \oplus C * A'^* * B')$ is a service matrix for the controlled road section, seen as a server with two inputs and two outputs, with $A' = \gamma^{(G/c)q_{max}\,\Delta x/v}\delta^{\Delta x/v}$, $B' = \left(\gamma^n \delta^{\Delta x/v+R} \quad e\right)$.

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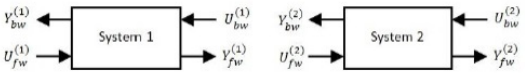
5. Connection of traffic systems



$$\begin{pmatrix} Y_{fw}^{(i)} \\ Y_{bw}^{(i)} \end{pmatrix} = \begin{pmatrix} (\beta^{(i)})_{11} & (\beta^{(i)})_{12} \\ (\beta^{(i)})_{21} & (\beta^{(i)})_{22} \end{pmatrix} \begin{pmatrix} U_{fw}^{(i)} \\ U_{bw}^{(i)} \end{pmatrix}, \qquad i = 1,2.$$

Theorem 4.

- A service matrix β for the whole system is given by:

$$\beta_{11} = \beta_{11}^{(2)}\beta_{11}^{(1)} \oplus \beta_{11}^{(2)}\beta_{12}^{(1)}\left(\beta_{21}^{(2)}\beta_{12}^{(1)}\right)^{*}\beta_{21}^{(2)}\beta_{11}^{(1)}$$
$$\beta_{12} = \beta_{11}^{(2)}\beta_{12}^{(1)}\left(\beta_{21}^{(2)}\beta_{12}^{(1)}\right)^{*}\beta_{22}^{(2)} \oplus \beta_{12}^{(2)}$$
$$\beta_{21} = \beta_{21}^{(1)} \oplus \beta_{22}^{(1)}\left(\beta_{21}^{(2)}\beta_{12}^{(1)}\right)^{*}\beta_{21}^{(2)}\beta_{11}^{(1)}$$
$$\beta_{22} = \beta_{22}^{(1)}\left(\beta_{21}^{(2)}\beta_{12}^{(1)}\right)^{*}\beta_{22}^{(2)}$$

such that

$$\begin{pmatrix} Y_{fw}^{(2)} \\ Y_{bw}^{(1)} \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} * \begin{pmatrix} U_{fw}^{(1)} \\ U_{bw}^{(2)} \end{pmatrix}.$$

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6. Roads and itineraries

- A road of m sections is obtained by composing m road sections.
- The service matrix of each road section can be obtained by Theorem 2, given fundamental traffic diagrams of each section.
- The service matrix of the whole road is then obtained by the composition of the road section systems and by applying Theorem 4.
- A controlled road of m sections is obtained similarly by composing $m-1$ uncontrolled road sections with one controlled road section.
- An itinerary in a controlled road network is build by composing a number of controlled roads.

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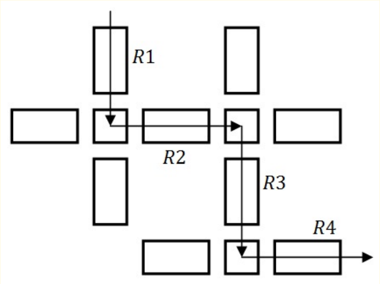

- A service matrix for the itinearay R1, 1 R2,R3, and R4:

    (1) Determine service matrices for all the uncontrolled sections of the itinerary, by Theorem 2.

    (2) Determine service matrices for all the controlled sections of the itinerary, by Theorem 3.

    (3) Determine service matrices for all the roads of the itinerary, by Theorem 4.

    (4) Determine a service matrix for the itinerary by connecting the systems R1, R2, R3, R4, by Theorem 4.

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6.1. Upper bound calculus.

- An arrival matrix is given expressing the traffic demand in the network.
- A service matrix is obtained as explained above.
- Theorem 1 gives upper bounds for the travel time for any input - output couple of the traffic system.

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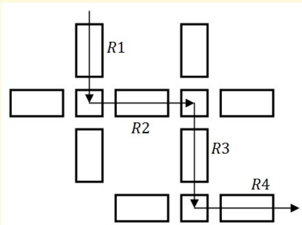
7. A numerical example



	R1	R2	R3	R4
Length Δx (meter)	150	150	100	100
Maximum flow q_{max} (veh/sec)	0.32	0.35	0.4	0.38
Initial density of cars n (veh/meter)	5/150	10/150	3/100	7/100
Cycle time (sec.)	60	90	80	-
Green time (sec.)	30	50	45	-

- $v = 15$ m/s, $w = -7$ m/s, $\rho_j = 1/10$ veh/m.
- U_{fw} of road 1, and U_{bw} of raod 4 are taken in such a way that the arrival flows do not exceed (in average) the service offered by the whole route.
- We first compute $T_{12} = 60$ s and $T_{21} = 8$ s.
- Then the arrival matrix is obtained.

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7.1. The results.

$$d_1 = \max(d_{11}, d_{12}) = \max(205{,}241) = 241 \text{ seconds.}$$

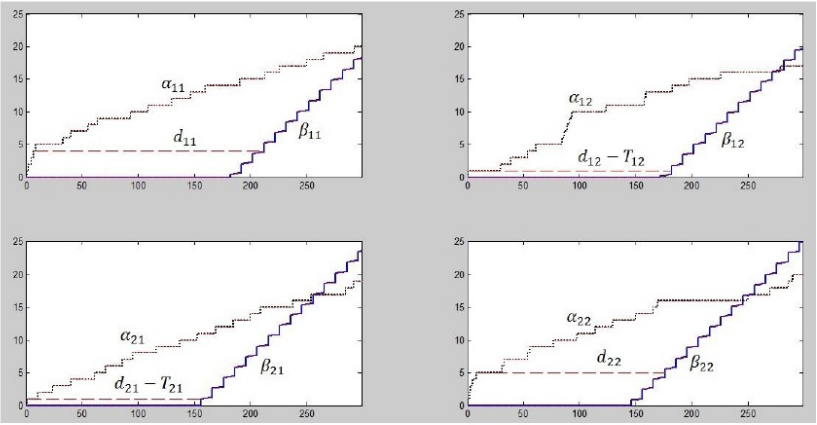

Figure 4. Arrival curves of the arrival matrix, service curves of the service matrix, and the time delays.

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Main references

- N. Farhi, H. Haj-Salem, and J.-P. Lebacque, *Upper bounds for the travel time on traffic systems.* EWGT 2014.
- N. Farhi, H. Haj-Salem, and J.-P. Lebacque, *Algebraic approach for performance bound calculus in transportation networks.* Transportation Research Record., 2013.
- C.-S. Chang, *Performance guarantees in communication networks.* Springer, 2000.
- J. .-Y. Le Boudec and P. Thiran, *Network calculus - A theory for deterministic queuing systems for the internet.* Springer-Verlag. , 2001.
- R. Cruz, *A calculus of network delay. Parts I and II*, vol. 1, no. 137 pp. 114-141, 1991.
- F. Baccelli, G. Cohen, G. J. Olsder, and J.-P. Quadrat, *Synchronization and Linearity.* Wiley, 1992.
- C. F. Daganzo, *The cell transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory.* Trans. Res. Part B: Methodological, 28(4), 269-287, 1994.

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