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Short communication

Stochastic resonance for an optimal detector with phase noise

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Abstract

A stochastic resonance effect, under the form of a noise-improved performance, is shown possible for an optimal detector. This is established with a nonlinear signal–noise mixture where the noise acts on the phase of a periodic signal. The optimal detector, achieving minimal probability of detection error, is explicitly derived. Conditions are exhibited where this minimal probability of error is reduced when the noise level is raised. These results contribute a new step in the investigation of stochastic resonance and in the inventory of its potentialities for nonlinear signal processing.

Keywords: Signal detection; Optimal detector; Stochastic resonance

1. Introduction

A nonlinear phenomenon, known as stochastic resonance, establishes that the transmission or the detectability of a signal buried in noise and processed by certain nonlinear systems, can be improved by raising the level of noise [4,8,16]. This counterintuitive phenomenon has gradually been reported for an increasing variety of signals, noises and nonlinear systems. It has been quantified by various measures such as signal-to-noise ratio [9,17], cross-correlation [5,6], mutual information [1,2], detection statistics [7,10], or other specific measures [13,18,19], all being shown improvable by raising the level of the noise, in definite conditions. These findings, progressively accumulating, have shed a new light on the status of noise in relation to nonlinear signal processing. Yet, so far, stochastic resonance has been shown possible only for *suboptimal* systems or detectors [3,12,20]. In each case where stochastic resonance was demonstrated, for a given measure of performance, noise improvement was possible only for the performance of suboptimal detectors or transmission systems; and if the optimal device was calculated, then its performance would undergo a monotonic degradation when raising the level of noise. Here, we show that noise improvement is also possible for the performance of an *optimal* detector, adding a new step in the development of stochastic resonance and in the inventory of its potentialities for nonlinear signal processing.

We place ourselves in the standard framework of statistical detection theory. One among two known signals $s_0(t)$ or $s_1(t)$ may be mixed to a noise $\eta(t)$, the resulting mixture forming the signal $x(t)$. This signal $x(t)$ is observed and it is to be decided whether $x(t)$ is formed by $s_0(t)$ mixed to the noise $\eta(t)$ (hypothesis H_0), or by $s_1(t)$ mixed to the noise $\eta(t)$ (hypothesis H_1). For this detection problem, we define the optimal detector as the one that minimizes the

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probability of detection error P_{er}, but a similar demonstration could be obtained with an optimal detector in the sense of Neyman–Pearson or of a Bayes cost function. The level of the noise $\eta(t)$ is quantified by its rms amplitude σ_η. The common expectation is that, as σ_η grows, the performance P_{er} of the optimal detector degrades. This expectation indeed prevails in standard detection problems, for instance for additive mixture with Gaussian noise. Yet, no theoretical proof guarantees that this expectation is true in generality. For nonlinear mixture with non-Gaussian noise, theoretical guarantees fail to be obtainable in generality. The fact is that this expectation is not generally true. We shall exhibit, for a nonlinear signal–noise mixture, an optimal detector whose performance can be improved by means of an increase in the level of noise.

2. Optimal detection theory

We briefly review the key elements of optimal detection theory, to make it clear, in a self-contained way, that they are valid in generality and especially for the detection problem with nonlinear signal-noise mixture we shall consider. Detailed expositions and applications can be found in [15,11].

For the general detection problem stated in Section 1, we further specify that hypothesis H_0 occurs with prior probability P_0, and H_1 with prior probability $P_1 = 1 - P_0$. The observed signal $x(t)$ is sampled at N distinct times t_j for $j = 1$ to N, so as to provide N data points $x_j = x(t_j)$. A given detector will decide hypothesis H_0 whenever the observation $\boldsymbol{x} = (x_1, \dots x_N)$ falls in the region $\mathscr{R}_0$ of $\mathbb{R}^N$, and it will decide H_1 when $\boldsymbol{x}$ falls in the complementary region $\mathscr{R}_1$ of $\mathbb{R}^N$. In doing so, the detector achieves an overall probability of detection error P_{er} expressable as

$$P_{er} = P_1 \int_{\mathscr{R}_0} p(\boldsymbol{x}|H_1)\,d\boldsymbol{x} + P_0 \int_{\mathscr{R}_1} p(\boldsymbol{x}|H_0)\,d\boldsymbol{x}, \quad (1)$$

where $p(\boldsymbol{x}|H_1)$ (respectively, $p(\boldsymbol{x}|H_0)$) is the probability density for observing $\boldsymbol{x}$ when H_1 (respectively, H_0) holds, and the notation $\int . d\boldsymbol{x}$ stands for the N-dimensional integral $\int \dots \int . dx_1 \dots dx_N$.

Since $\mathscr{R}_0$ and $\mathscr{R}_1$ are complementary in $\mathbb{R}^N$, one has

$$\int_{\mathscr{R}_0} p(\boldsymbol{x}|H_1)\,d\boldsymbol{x} = 1 - \int_{\mathscr{R}_1} p(\boldsymbol{x}|H_1)\,d\boldsymbol{x}, \quad (2)$$

which substituted in Eq. (1) yields

$$P_{er} = P_1 + \int_{\mathscr{R}_1} [P_0\, p(\boldsymbol{x}|H_0) - P_1\, p(\boldsymbol{x}|H_1)]\,d\boldsymbol{x}. \quad (3)$$

The detector that minimizes P_{er} can be obtained by making the integral over $\mathscr{R}_1$ in the right-hand side of Eq. (3) the more negative possible. This is realized by including into $\mathscr{R}_1$ all and only those points $\boldsymbol{x}$ for which the integrand $P_0\, p(\boldsymbol{x}|H_0) - P_1\, p(\boldsymbol{x}|H_1)$ is negative. This gives the optimal detector, also known as the maximum a posteriori probability (MAP) detector, which implements the test

$$L(\boldsymbol{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P_0}{P_1}, \quad (4)$$

through the use of the likelihood ratio

$$L(\boldsymbol{x}) = \frac{p(\boldsymbol{x}|H_1)}{p(\boldsymbol{x}|H_0)}. \quad (5)$$

When the values accessible to the observation $\boldsymbol{x}$, instead of being continuously distributed, are restricted to a set of discrete values $\boldsymbol{x}_n$, the same formalism applies with probability densities defined in terms of Dirac delta functions as $p(\boldsymbol{x}|H_i) = \sum_n \Pr\{\boldsymbol{x}_n|H_i\}\,\delta(\boldsymbol{x} - \boldsymbol{x}_n)$, $i = 0,\ 1$, and in this case the likelihood ratio is defined only at these points $\boldsymbol{x}_n$ as

$$L(\boldsymbol{x}_n) = \frac{\Pr\{\boldsymbol{x}_n|H_1\}}{\Pr\{\boldsymbol{x}_n|H_0\}}. \quad (6)$$

The minimal P_{er} reached by the MAP detector of Eq. (4) is expressable as

$$P_{er} = \int_{\mathbb{R}^N} \min[P_0\, p(\boldsymbol{x}|H_0), P_1\, p(\boldsymbol{x}|H_1)]\,d\boldsymbol{x}. \quad (7)$$

Since $\min(a, b) = (a + b - |a - b|)/2$, the minimal probability of error of Eq. (7) reduces to

$$P_{er} = \frac{1}{2} - \frac{1}{2} \int_{\mathbb{R}^N} |P_1\, p(\boldsymbol{x}|H_1) - P_0\, p(\boldsymbol{x}|H_0)|\,d\boldsymbol{x}. \quad (8)$$

3. Nonlinear signal–noise mixture

We now introduce a specific detection problem, amenable to the general treatment of Section 2. We consider a periodic wave $w(t)$ of period unity. A possibility could be $w(t)=\sin(2\pi t)$, but $w(t)$ will be further specified later. One of the two signals to be detected is the wave $w(t)$ with frequency ν_0, i.e. $s_0(t)=w(\nu_0 t)$ (prior probability P_0); the other signal is the same wave $w(t)$ with frequency $\nu_1 \neq \nu_0$, i.e. $s_1(t)=w(\nu_1 t)$ (prior probability P_1). The noise $\eta(t)$ acts on signals $s_0(t)$ and $s_1(t)$ as a phase noise, so as to form the observable signal

$$x(t)=w[\nu_0 t+\eta(t)] \quad \text{(hypothesis H}_0\text{)},$$

or

$$x(t)=w[\nu_1 t+\eta(t)] \quad \text{(hypothesis H}_1\text{)}.$$

Such periodic signals corrupted by a phase noise arise, for instance, when a periodic wave propagates in a fluctuating medium or through a fluctuating interface. Phase noise is naturally present in oscillators and phase-locked loops. Also, the conditions considered may find applicability in phase-contrast microscopy or coherent imaging. A simple concretization of the present setting is provided by a plane wave radiated and/or received by transducers subjected to random motions producing the phase noise.

Based on the data set $\boldsymbol{x}=(x_1,\dots,x_N)$ we want to decide between hypotheses H_0 or H_1, i.e. to detect whether the wave corrupted by the phase noise has frequency ν_0 or ν_1.

We consider $\eta(t)$ to be a white noise, i.e. $\eta(t_j)$ and $\eta(t_k)$ are statistically independent for any two sampling times $t_j \neq t_k$. Then, the conditional probabilities defined in Section 2 factorize as $\Pr\{\boldsymbol{x}|\nu_1\} = \prod_{j=1}^{N} \Pr\{x_j|\nu_1\}$ and $\Pr\{\boldsymbol{x}|\nu_0\}=\prod_{j=1}^{N} \Pr\{x_j|\nu_0\}$. Also, $\eta(t)$ is assumed stationary, with cumulative distribution function $F_\eta(u)$ and probability density function $f_\eta(u)=\mathrm{d}F_\eta/\mathrm{d}u$.

In order to allow a complete analytical treatment of the optimal detector, we consider the simple case where $w(t)$ is a square wave of period 1 with $w(t)=1$ when $t\in[0,1/2)$ and $w(t)=-1$ when $t\in[1/2,1)$. We then have the probabilities

$$\Pr\{x_j=1|\nu_1\}$$

$$=\Pr\{w[\nu_1 t_j+\eta(t_j)]=1\} \tag{9}$$

$$=\Pr\left\{\nu_1 t_j+\eta(t_j)\in\bigcup_k [k,k+1/2)\right\} \tag{10}$$

$$=\Pr\left\{\eta(t_j)\in\bigcup_k [k-\nu_1 t_j,k-\nu_1 t_j+1/2)\right\} \tag{11}$$

$$=\sum_{k=-\infty}^{+\infty}\int_{k-\nu_1 t_j}^{k-\nu_1 t_j+1/2} f_\eta(u)\,\mathrm{d}u=\mathscr{F}(\nu_1 t_j) \tag{12}$$

with the function

$$\mathscr{F}(u)=\sum_{k=-\infty}^{+\infty}[F_\eta(k-u+1/2)-F_\eta(k-u)], \tag{13}$$

k integer, and

$$\Pr\{x_j=-1|\nu_1\}=1-\Pr\{x_j=1|\nu_1\}. \tag{14}$$

In the same way, we have

$$\Pr\{x_j=1|\nu_0\}=\mathscr{F}(\nu_0 t_j) \tag{15}$$

and

$$\Pr\{x_j=-1|\nu_0\}=1-\Pr\{x_j=1|\nu_0\}. \tag{16}$$

Given P_0 and $F_\eta(u)$, when a realization of $\boldsymbol{x}$ is observed, Eqs. (12)–(16) allow an explicit evaluation of the likelihood ratio of Eq. (6) under the form $L(\boldsymbol{x})=(\prod_{j=1}^{N} \Pr\{x_j|\nu_1\})/(\prod_{j=1}^{N} \Pr\{x_j|\nu_0\})$, making possible an explicit implementation of the optimal MAP detector of Eq. (4). This optimal detector achieves the minimal probability of error, which, according to Eq. (8), is explicitly computable as

$$P_{\mathrm{er}}=\frac{1}{2}-\frac{1}{2}\sum_{x_1=-1}^{1}\cdots\sum_{x_N=-1}^{1}|P_1\Pr\{x_1|\nu_1\}\dots\Pr\{x_N|\nu_1\}-P_0\Pr\{x_1|\nu_0\}\dots\Pr\{x_N|\nu_0\}|, \tag{17}$$

the multiple sum running over the 2^N possible states for the data $\boldsymbol{x}$.

4. Noise-enhanced optimal detection

We now exhibit conditions where the performance of the optimal detector measured by P_{er} of Eq. (17) can be improved when the noise rms amplitude σ_η grows.

For illustration, we first consider the case where $\eta(t)$ is chosen in the class of generalized Gaussian noises, defined by the standardized density

$$f_{gg}(u) = A\exp(-|bu|^\alpha) \tag{18}$$

with $A = (\alpha/2)[\Gamma(3/\alpha)]^{1/2}/[\Gamma(1/\alpha)]^{3/2}$ and $b = [\Gamma(3/\alpha)/\Gamma(1/\alpha)]^{1/2}$, parameterized by the positive exponent α. Such generalized Gaussian noise models are widely used in ocean acoustics and sonar applications for instance [14]. The density of $\eta(t)$ is then taken as $f_\eta(u) = f_{gg}(u/\sigma_\eta)/\sigma_\eta$, and our finding its that, for any $\alpha > 2$, the probability of error P_{er} of the optimal detector undergoes a nonmonotonic evolution as σ_η is raised, instead of a monotonic increase. This is illustrated by Fig. 1 which represents the evolution of P_{er} of Eq. (17) as a function of σ_η, for different α. For the theoretical evaluations of P_{er} in Figs. 1 and 2, the infinite sums of Eqs. (12) or (13) have been truncated by considering the zero-mean densities $f_\eta(u)$ to be negligible outside the interval $[-6\sigma_\eta, 6\sigma_\eta]$, which provides very good approximation. With standard Gaussian noise ($\alpha = 2$), P_{er} is found to monotonically increase as σ_η grows, but for $\alpha > 2$, Fig. 1 shows ranges of σ_η where P_{er} decreases as σ_η grows. This demonstrates the possibility of improving the performance of the optimal detector by raising the level of a generalized Gaussian noise with $\alpha > 2$. This also shows that the effect is robust with respect to changes in the type of the probability density, since it is qualitatively preserved when $\alpha > 2$ is varied.

The improvement as a reduction of P_{er} can be found larger if one moves to other classes of densities for $\eta(t)$. Consider the class of Gaussian mixture with standardized density ($0 < m < 1$)

$$f_{gm}(u) = \frac{1}{2\sqrt{2\pi}\sqrt{1-m^2}}\left\{\exp\left[-\frac{(u+m)^2}{2(1-m^2)}\right] + \exp\left[-\frac{(u-m)^2}{2(1-m^2)}\right]\right\}. \tag{19}$$

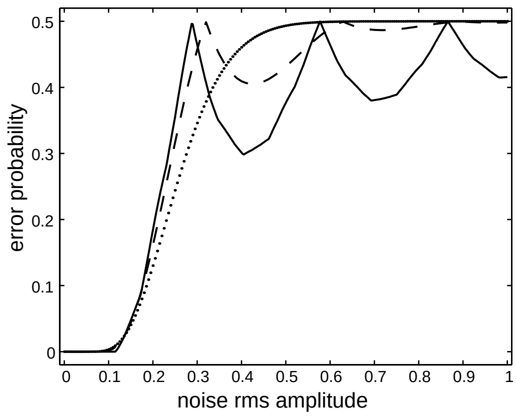

Fig. 1. Probability of detection error P_{er} of the optimal detector, as a function of the rms amplitude σ_η of the zero-mean noise $\eta(t)$ chosen Gaussian (dotted line), generalized Gaussian with $\alpha = 4$ (dashed), uniform (solid). Also $P_0 = 0.5$, $v_0 = 1$, $v_1 = 2/3$, $N = 11$ data samples equispaced with time step 0.2 from $t_1 = 0$ to $t_{11} = 2$.

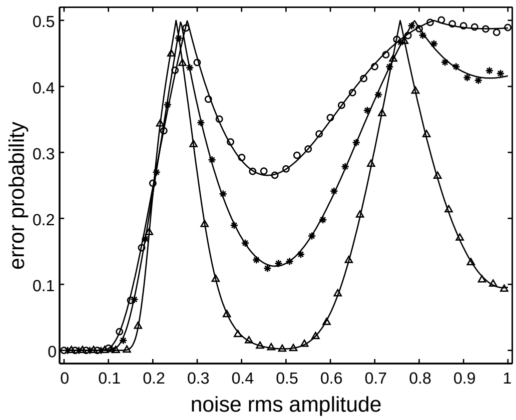

Fig. 2. Probability of detection error P_{er} of the optimal detector, as a function of the rms amplitude σ_η of the Gaussian-mixture noise $\eta(t)$ with density $f_{gm}(u/\sigma_\eta)/\sigma_\eta$ from Eq. (19). The solid lines are P_{er} from Eq. (17); the discrete points are P_{er} numerically estimated from 10^4 Monte Carlo trials of the MAP test for each σ_η; with (○) $m = 0.9$, (∗) $m = 0.95$, (△) $m = 0.99$. Also $P_0 = 0.5$, $v_0 = 1$, $v_1 = 2/3$, $N = 6$ data samples equispaced with time step 0.3 from $t_1 = 0$ to $t_6 = 1.5$.

With $f_\eta(u) = f_{gm}(u/\sigma_\eta)/\sigma_\eta$, Fig. 2 shows again conditions of nonmonotonic evolutions of P_{er} as σ_η grows, with possibilities of decreasing P_{er} by increasing σ_η. Fig. 2 also offers numerical validations of the theoretical performance, through Monte Carlo implementation of the optimal detector of Eq. (4).

5. Discussion

The present results essentially stand for a demonstration in principle of the feasibility of a stochastic resonance effect in optimal processing.

A qualitative explanation of the effect in optimal detection here can be that the phase noise, at a sufficient level, is able to bring some shift between $w[\nu_0 t + \eta(t)]$ and $w[\nu_1 t + \eta(t)]$ that can make these signals, on average, more distinguishable, whence a reduced $P_{\rm er}$. Of course in the present setting $P_{\rm er}$ is always zero at zero noise (a common behavior of any reasonable detection scheme), and then a pre-existing non-optimal amount of phase noise has to be present in order to have a possibility of improvement through noise enhancement.

The optimal detector, as implemented in Figs. 1 and 2, is a coherent or synchronized detector, which has access to a time origin where the waveform $w(\nu_0 t)$ or $w(\nu_1 t)$ starts a rising front, and where the detector will start its sampling sequence $\{t_j\}$, $j = 1$ to N. Yet, it can easily be verified from Eq. (17) that there is little change in $P_{\rm er}$ if the sampling sequence $\{t_j\}$ does not start on a rising front, provided $\{t_j\}$ realizes a sufficiently even or representative coverage of the periodic waves $w(\nu_0 t)$ and $w(\nu_1 t)$ in relation to their periods $1/\nu_0$ and $1/\nu_1$. Especially, the important property of a nonmonotonic $P_{\rm er}$ is preserved with sequences $\{t_j\}$ that do not start on a rising front. The performance of an incoherent or asynchronous detector could be obtained by an additional averaging on $P_{\rm er}$ of Eq. (17) performed over a random initial time for the sampling sequence $\{t_j\}$, and nonmonotonic evolutions would still exist for this performance. Detailed study of the influence of the sampling sequence remains open for future work.

For practical use of the reported effect, one needs to be able to increase the noise level while controlling its nature, and especially its probability density. Such issues are not explicitly addressed here, and may require evolutions in the setting and conditions considered here. Again, the focus of the present paper is more at the conceptual level of a proof of feasibility in principle. This contributes to the inventory of the various forms, aspects and potentialities of the still emerging effect of stochastic resonance, which ultimately may lead to useful benefits for nonlinear signal processing.

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