Noise-Enhanced Performance for an Optimal Bayesian Estimator

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***Abstract*—A novel instance of a stochastic resonance effect, under the form of a noise-improved performance, is shown to be possible for an optimal Bayesian estimator. Estimation of the frequency of a periodic signal corrupted by a phase noise is considered. The optimal Bayesian estimator, achieving the minimum of the mean square estimation error, is explicitly derived. Conditions are exhibited where this minimal error is reduced when the noise level is raised, over some ranges, where this occurs essentially with non-Gaussian noise, in the tested configurations. These results contribute a new step in the exploration of stochastic resonance and its potentialities for signal processing.**

***Index Terms*—Bayesian estimation, optimal estimator, stochastic resonance.**

I. Introduction

STOCHASTIC resonance, in a general sense, can be described as a phenomenon by which some processing done on a signal can benefit from the presence of noise [1], [2]. This counterintuitive phenomenon has essentially been reported in nonlinear settings and with various types of signals and noises [3]–[5]. Instances have been observed in electronic circuits [6], [7], optical devices [8], [9], magnetic systems [10], [11], and neural processes [12], [13]. In each case, a measure of performance is considered that quantifies the efficacy of some processing on the signal in the presence of noise. Stochastic resonance is then characterized by the possibility of conditions where an increase in the level of the noise results in an improved performance. Examples have been studied of nonlinear transmission systems with an output signal-to-noise ratio (SNR) that is improvable by means of an increase in the input noise [3], [14], information channels with a transinformation or a capacity that can be increased when the noise over the channel is enhanced [15], [16], or nonlinear lines where propagation conditions improve when the noise is raised [17], [18]. In addition, detection or estimation problems have been studied on nonlinear signal-noise mixtures, where the efficacy improves when the noise increases [19]–[23].

Investigations on stochastic resonance have often been conducted with Gaussian noise [3], [7], [14], [21]; beyond its practical importance, Gaussian noise is a case that very often can be worked out, especially analytically, in the most extended way. However, non-Gaussian noise is also frequently met in pratice, especially in nonlinear environments, and stochastic resonance has also been obtained with non-Gaussian noise [3], [16], [24]–[26]. Thus far, instances of stochastic resonance have been observed with Gaussian noise and others with non-Gaussian noise. This largely depends on the specific setting, and especially on the type of the nonlinear signal-noise coupling, on the nature of the information signal, and on the measure of performance receiving improvement from the noise. As we will see, the novel instance of stochastic resonance we will present here essentially takes place with non-Gaussian noise in the explored configurations.

The progressive development of all these studies on stochastic resonance has disclosed many configurations and forms under which it can occur. Yet, so far, stochastic resonance has essentially been reported for suboptimal devices or processors [27], [28], [23]. In each case where stochastic resonance was demonstrated, for a given measure of performance, noise improvement was possible only for the performance of suboptimal processors, and if the optimal processor was calculated, then its performance would undergo a monotonic degradation when raising the level of noise.

The present study enlarges the conditions of applicability of stochastic resonance. It investigates conditions of optimal processing in a Bayesian estimation problem and demonstrates the possibility of improving the performance of an optimal estimator by operating at higher noise levels. The addressed problem is the estimation of the frequency of a periodic signal in the presence of a nonlinear signal-noise mixture where the noise acts on the phase of the signal.

II. Optimal Bayesian Estimation

We briefly review the essential elements of optimal Bayesian estimation, to make it clear, in a self-contained way, that they are valid in generality and especially for the estimation problem with the nonlinear signal-noise mixture that we will address. Detailed expositions and applications can be found in [29] and [30].

Observation of a random signal $x(t)$ at N different times t_j for $j = 1$ to N provides N data points $x_j = x(t_j)$. This signal $x(t)$ is dependent on a parameter ν, whose possible values are distributed according to the prior probability density function (pdf) $p_\nu(u)$. In order to estimate the value of ν that produced

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the observed data $\boldsymbol{x} = (x_1, \dots x_N)$, an estimator $\hat{\nu}(\boldsymbol{x})$ is constructed. Once $\boldsymbol{x}$ is observed, a posterior pdf $p(\nu \,|\, \boldsymbol{x})$ for the parameter ν can be defined. A mean square error in the estimation follows as the expectation (conditioned by observation $\boldsymbol{x}$)

$$\mathcal{E}(\boldsymbol{x}) = \mathrm{E}[(\nu - \hat{\nu})^2 \,|\, \boldsymbol{x}] = \int [\nu - \hat{\nu}(\boldsymbol{x})]^2 p(\nu \,|\, \boldsymbol{x})\, d\nu. \tag{1}$$

It is easy to show that $\mathcal{E}(\boldsymbol{x})$ of (1) can equivalently be expressed as

$$\mathcal{E}(\boldsymbol{x}) = [\hat{\nu} - \mathrm{E}(\nu \,|\, \boldsymbol{x})]^2 + \mathrm{var}(\nu \,|\, \boldsymbol{x}), \tag{2}$$

with $\mathrm{E}(\nu \,|\, \boldsymbol{x}) = \int \nu p(\nu \,|\, \boldsymbol{x})\, d\nu$, and $\mathrm{var}(\nu \,|\, \boldsymbol{x}) = \int [\nu - \mathrm{E}(\nu \,|\, \boldsymbol{x})]^2 p(\nu \,|\, \boldsymbol{x})\, d\nu$.

Since $\mathrm{var}(\nu \,|\, \boldsymbol{x})$ in (2) is non-negative and independent of $\hat{\nu}$, the optimal Bayesian estimator that minimizes error $\mathcal{E}(\boldsymbol{x})$, for any given observation $\boldsymbol{x}$, comes out as

$$\hat{\nu}_{\mathrm{B}}(\boldsymbol{x}) = \mathrm{E}(\nu \,|\, \boldsymbol{x}) = \int \nu p(\nu \,|\, \boldsymbol{x})\, d\nu \tag{3}$$

and its performance is measured by the minimal error

$$\mathcal{E}_{\mathrm{B}}(\boldsymbol{x}) = \mathrm{var}(\nu \,|\, \boldsymbol{x}) = \int [\nu - \mathrm{E}(\nu \,|\, \boldsymbol{x})]^2 p(\nu \,|\, \boldsymbol{x})\, d\nu. \tag{4}$$

A model of how the observation $\boldsymbol{x}$ is produced in relation to the parameter ν (and also to the noise spoiling the observation) allows one to define the pdf $p(\boldsymbol{x} \,|\, \nu)$ of observing $\boldsymbol{x}$ given ν. With the prior information summarized by $p_\nu(\nu)$, the Bayes rule then provides access to the posterior pdf under the form

$$p(\nu \,|\, \boldsymbol{x}) = \frac{p(\boldsymbol{x} \,|\, \nu) p_\nu(\nu)}{p(\boldsymbol{x})} \tag{5}$$

with the pdf $p(\boldsymbol{x}) = \int p(\boldsymbol{x} \,|\, \nu) p_\nu(\nu) d\nu$.

For any given observation $\boldsymbol{x}$, the optimal Bayesian estimator $\hat{\nu}_{\mathrm{B}}(\boldsymbol{x})$ from (3) achieves the minimum $\mathcal{E}_{\mathrm{B}}(\boldsymbol{x})$ from (4) of the error $\mathcal{E}(\boldsymbol{x})$ from (1). Consequently, $\hat{\nu}_{\mathrm{B}}(\boldsymbol{x})$ also achieves the minimum $\bar{\mathcal{E}}_{\mathrm{B}}$ of error $\mathcal{E}(\boldsymbol{x})$ averaged over every possible observation $\boldsymbol{x}$, i.e., $\hat{\nu}_{\mathrm{B}}(\boldsymbol{x})$ minimizes $\int \mathcal{E}(\boldsymbol{x}) p(\boldsymbol{x})\, d\boldsymbol{x}$, and the minimum that is reached is

$$\bar{\mathcal{E}}_{\mathrm{B}} = \int \mathrm{var}(\nu \,|\, \boldsymbol{x}) p(\boldsymbol{x})\, d\boldsymbol{x} \tag{6}$$

where $\int \cdot d\boldsymbol{x}$ stands for the N-dimensional integral $\int \cdots \int \cdot dx_1 \cdots dx_N$.

We now address a specific estimation problem, in which the observation $\boldsymbol{x}$ incorporates the influence of a corrupting noise. In standard situations, i.e., additive signal-noise mixture or Gaussian noise, the optimal estimator of (3) essentially has a performance that is measured by (4) or (6), which degrades monotonically when the noise level is raised. Here, we will show, with a nonlinear signal-noise mixture and essentially non-Gaussian noise, that it is possible to have an optimal Bayesian estimator whose performance can be improved by raising the level of the noise.

III. Estimation With Phase Noise

We consider a periodic wave $w(\nu t)$ of (unknown) frequency ν, where $w(t)$ is a periodic waveform of period unity. A possibility could be $w(t) = \sin(2\pi t)$, but $w(t)$ will be further specified later. A noise $\eta(t)$ acts on the phase of the wave to form the observable signal

$$x(t) = w[\nu t + \eta(t)]. \tag{7}$$

Such a periodic signal corrupted by a phase noise arises, for instance, when a periodic wave propagates in a fluctuating medium or through a fluctuating interface. Phase noise is naturally present in oscillators, phase-locked loops, and coherent imaging [31]–[34]. A simple concretization of the present setting is provided by a plane wave radiated or received by a transducer subjected to a random motion producing the phase noise.

Based on the data $\boldsymbol{x} = (x_1, \dots x_N)$ observed on the noisy signal $x(t)$, the frequency ν is to be estimated.

We consider the noise samples $\eta(t_j)$ statistically independent for distinct t_j's so that the conditional pdf $p(\boldsymbol{x} \,|\, \nu)$ introduced in Section II factorizes as $p(\boldsymbol{x} \,|\, \nu) = \prod_{j=1}^N p(x_j \,|\, \nu)$. In addition, the samples $\eta(t_j)$ are identically distributed, with cumulative distribution function $F_\eta(u)$ and probability density function $f_\eta(u) = dF_\eta/du$.

In order to allow a complete analytical treatment of the optimal Bayesian estimator, we consider the simple case where $w(t)$ is a square wave of period 1 with $w(t) = 1$ when $t \in [0, 1/2)$ and $w(t) = -1$ when $t \in [1/2, 1)$. With $\delta(\cdot)$ as the Dirac delta function, we have the pdf

$$\begin{aligned} p(x_j \,|\, \nu) = \Pr\{x_j = -1 \,|\, \nu\} \delta(x_j + 1) \\ + \Pr\{x_j = 1 \,|\, \nu\} \delta(x_j - 1) \end{aligned} \tag{8}$$

with the probability

$$\Pr\{x_j = 1 \,|\, \nu\} = \Pr\{w[\nu t_j + \eta(t_j)] = 1\} \tag{9}$$

$$= \Pr\left\{ \nu t_j + \eta(t_j) \in \bigcup_k [k, k + 1/2) \right\} \tag{10}$$

$$= \Pr\left\{ \eta(t_j) \in \bigcup_k [k - \nu t_j, k - \nu t_j + 1/2) \right\} \tag{11}$$

$$= \sum_{k=-\infty}^{+\infty} \int_{k-\nu t_j}^{k-\nu t_j + 1/2} f_\eta(u)\, du \tag{12}$$

$$= \sum_{k=-\infty}^{+\infty} [F_\eta(k - \nu t_j + 1/2) - F_\eta(k - \nu t_j)] \tag{13}$$

where k is an integer, and the probability

$$\Pr\{x_j = -1 \,|\, \nu\} = 1 - \Pr\{x_j = 1 \,|\, \nu\}. \tag{14}$$

The pdf $p(\boldsymbol{x} \,|\, \nu) = \prod_{j=1}^N p(x_j \,|\, \nu)$, according to (8), will involve products of quantities of the form $\Pr\{x_j = \pm 1 | \nu\} \delta(x_j \mp 1)$. The posterior pdf of (5) is then expressable as

$$p(\nu | \boldsymbol{x}) = \frac{p_\nu(\nu) \prod_{j=1}^N \Pr\{x_j \,|\, \nu\}}{\int_{-\infty}^{+\infty} p_\nu(\nu) \prod_{j=1}^N \Pr\{x_j \,|\, \nu\}\, d\nu} \tag{15}$$

in which expression the data vector $\boldsymbol{x} = (x_1, \dots x_N)$ is now limited to the 2^N possible states of the form $(x_1 = \pm 1, \dots x_N = \pm 1)$. Equation (15) is obtained as the Dirac delta functions introduced by (8) disappear by simplification between the numerator and denominator of (5). This simplification applies since the Dirac pulses are located at the same values in the numerator and denominator.[1]

Equations (13) and (14) allow an explicit evaluation of the probabilities $\Pr\{x_j \,|\, \nu\}$ for $x_j = \pm 1$, as a function of the properties of the noise $\eta(t)$ conveyed by $F_\eta(u)$. These probabilities $\Pr\{x_j \,|\, \nu\}$ are all that is needed to provide access to the conditional pdf of (15), which opens the way to an explicit calculation (possibly through numerical integration) of the optimal Bayesian estimate from (3) and to the performance of this estimation measured by (4) or (6).

Explicitly, $\mathcal{E}_{\rm B}(\boldsymbol{x})$ of (4), which is a function of the observation $\boldsymbol{x}$, is computable as

$$\mathcal{E}_{\rm B}(\boldsymbol{x}) = \int_{-\infty}^{+\infty} \nu^2 p(\nu \,|\, \boldsymbol{x})\, d\nu - \left[\int_{-\infty}^{+\infty} \nu p(\nu \,|\, \boldsymbol{x})\, d\nu\right]^2 \quad (16)$$

and its average $\bar{\mathcal{E}}_{\rm B}$ over $\boldsymbol{x}$ of (6) comes out [the Dirac delta functions introduced by (8) disappear by integration] as

$$\bar{\mathcal{E}}_{\rm B} = \sum_{x_1 \in \{-1,1\}} \cdots \sum_{x_N \in \{-1,1\}} \mathcal{E}_{\rm B}(\boldsymbol{x}) \times \int_{-\infty}^{+\infty} p_\nu(\nu) \prod_{j=1}^{N} \Pr\{x_j \,|\, \nu\}\, d\nu \quad (17)$$

where the multiple sum runs over the 2^N possible states for the data $\boldsymbol{x}$.

IV. Noise-Enhanced Optimal Estimation

We now exhibit conditions where the performance of the optimal estimator measured by $\bar{\mathcal{E}}_{\rm B}$ of (17) can be improved when the noise rms amplitude σ_η grows.

For illustration, we consider the case where the frequency ν to be estimated is distributed according to a Gaussian prior pdf $p_\nu(u)$ with mean m_ν and standard deviation σ_ν. In addition, $\eta(t)$ is chosen in the class of generalized Gaussian noises defined by the standardized pdf

$$f_{\rm gg}(u) = A \exp(-|bu|^\alpha) \quad (18)$$

with $A = (\alpha/2)[\Gamma(3/\alpha)]^{1/2}/[\Gamma(1/\alpha)]^{3/2}$, and $b = [\Gamma(3/\alpha)/\Gamma(1/\alpha)]^{1/2}$ parameterized by the positive exponent α. For $\alpha = 2$, one recovers the Gaussian density; for $0 < \alpha < 2$, one obtains leptokurtic densities with tails thicker than the Gaussian; for $2 < \alpha < +\infty$, one gets platikurtic densities with tails thinner than the Gaussian, up to $\alpha = +\infty$, yielding the uniform density.

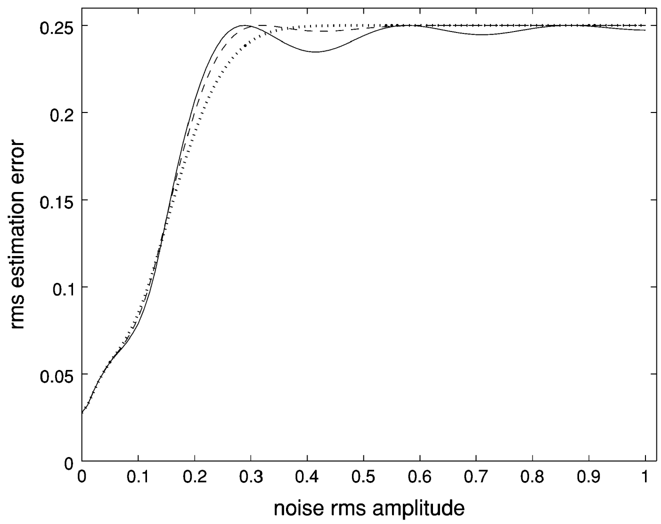


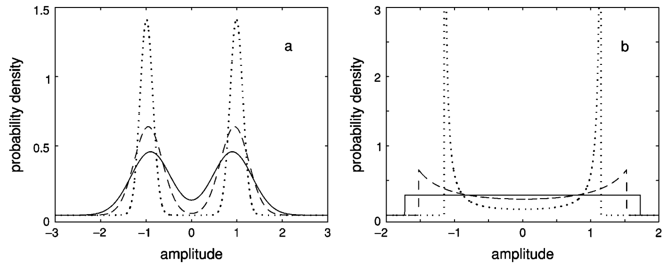
Fig. 1. Rms error $\bar{\mathcal{E}}_{\rm B}^{1/2}$ from (17) of the optimal estimator as a function of the rms amplitude σ_η of the zero-mean noise $\eta(t)$ chosen Gaussian (dotted line), generalized Gaussian with $\alpha = 4$ (dashed), uniform (solid). Prior pdf $p_\nu(u)$ is Gaussian with $m_\nu = 1$ and $\sigma_\nu = 0.25$ and $N = 14$ data samples equispaced with time step 0.075 from $t_1 = 0$ to $t_{14} = 1$.

The pdf of $\eta(t)$ is then taken as $f_\eta(u) = f_{\rm gg}(u/\sigma_\eta)/\sigma_\eta$. Fig. 1 represents the rms error $\bar{\mathcal{E}}_{\rm B}^{1/2}$ from (17) of the optimal estimator, as a function of the noise rms amplitude σ_η, for different α. The standard expectation with a Bayesian estimator of ν is that error $\bar{\mathcal{E}}_{\rm B}^{1/2}$ goes to σ_ν as $\sigma_\eta \to +\infty$, and $\bar{\mathcal{E}}_{\rm B}^{1/2}$ starts below σ_ν when $\sigma_\eta \to 0$. Our point will be that such an evolution of $\bar{\mathcal{E}}_{\rm B}^{1/2}$, as σ_η grows, is not necessarily monotonically increasing, but can be nonmonotonic. In Fig. 1, we observe that with Gaussian noise $(\alpha = 2)$, the estimation error $\bar{\mathcal{E}}_{\rm B}^{1/2}$ monotonically increases as σ_η grows. However, as one departs from Gaussian noise with $\alpha > 2$, error $\bar{\mathcal{E}}_{\rm B}^{1/2}$ comes to experience a nonmonotonic evolution, with ranges of σ_η, where $\bar{\mathcal{E}}_{\rm B}^{1/2}$ decreases as σ_η grows. This possibility of lowering $\bar{\mathcal{E}}_{\rm B}^{1/2}$ by increasing σ_η gets more pronounced as α increases toward $+\infty$. Although the effect remains modest in Fig. 1, this is an effective demonstration[2] of the feasibility of improving the performance of the optimal estimator by raising the level of a generalized Gaussian noise with $\alpha > 2$, which is a novel form of stochastic resonance.

In Fig. 1, we observe that $\bar{\mathcal{E}}_{\rm B}^{1/2}$ first starts to rise as σ_η increases above zero. A similar behavior of $\bar{\mathcal{E}}_{\rm B}^{1/2}$ around the origin will also be observed later in the stochastic resonance of Figs. 3 and 4. Such a behavior means that in the signal-noise mixture, a nonzero minimal amount of noise has to pre-exist in order to have access to a range of σ_η, where $\bar{\mathcal{E}}_{\rm B}^{1/2}$ starts to diminish. The primary important finding we want to emphasize here is the existence of some ranges of the noise level where $\bar{\mathcal{E}}_{\rm B}^{1/2}$ decreases as σ_η grows, which is an *a priori* unexpected benefit brought in by the noise. Later on, however, in the stochastic resonance of Figs. 5–7, we will additionally show the possibility of an error

[1]This mode of operation, relying on Dirac delta functions, allows a uniform treatment that is equally applicable for both continuous and discrete data. Alternatively, the whole Bayesian estimation framework of Section II could be rewritten separately, from the beginning, with discrete probabilities $\Pr\{x_j \,|\, \nu\}$ instead of continuous densities $p(x_j \,|\, \nu)$ and no Dirac delta functions and would ultimately lead to the same (15) for discrete data.

[2]In Fig. 1, for $\alpha = 4$, to have access to the cumulative distribution $F_{\rm gg}(u) = (1/2) + \int_0^u f_{\rm gg}(v)\, dv$, we used the Maple mathematical software for high-accuracy numerical evaluation of this definite integral to construct an analytical approximation of $F_{\rm gg}(u)$ with a relative accuracy better than 10^{-4} and based on a rational function approximation for small $|u|$'s, and on an asymptotic expansion for large $|u|$'s, following an approach much similar to that used in [35].



(a) (b)

Fig. 2. Standardized pdf. (a) $f_{\rm gm}(u)$ of (19) with $m = 0.9$ (solid line), $m = 0.95$ (dashed line), and $m = 0.99$ (dotted line). (b) $f_{\rm sq}(u)$ of (22) with $\beta = 0.1$ (solid line), $\beta = 1$ (dashed line), and $\beta = 5$ (dotted line).

$\bar{\mathcal{E}}_{\rm B}^{1/2}$ that starts to decrease at the origin, as soon as σ_η grows above zero, which is another aspect of the benefit brought in by the noise.

The improvement visible in Fig. 1 as a reduction of $\bar{\mathcal{E}}_{\rm B}^{1/2}$ can be found larger if one moves to other classes of pdf for the noise $\eta(t)$. Consider the class of Gaussian mixture with standardized pdf $(0 < m < 1)$

$$f_{\rm gm}(u) = \frac{1}{2\sqrt{2\pi}\sqrt{1-m^2}}\left\{\exp\left[-\frac{(u+m)^2}{2(1-m^2)}\right] + \exp\left[-\frac{(u-m)^2}{2(1-m^2)}\right]\right\} \quad (19)$$

and cumulative distribution function

$$F_{\rm gm}(u) = \frac{1}{2} + \frac{1}{4}\left[\operatorname{erf}\left(\frac{u+m}{\sqrt{2}\sqrt{1-m^2}}\right) + \operatorname{erf}\left(\frac{u-m}{\sqrt{2}\sqrt{1-m^2}}\right)\right]. \quad (20)$$

Some examples of the pdf $f_{\rm gm}(u)$ of (19), for different m, are plotted in Fig. 2(a).

With $f_\eta(u) = f_{\rm gm}(u/\sigma_\eta)/\sigma_\eta$ and $F_\eta(u) = F_{\rm gm}(u/\sigma_\eta)$, Fig. 3 again shows conditions of nonmonotonic evolutions of $\bar{\mathcal{E}}_{\rm B}^{1/2}$ as σ_η grows, with possibilities of decreasing $\bar{\mathcal{E}}_{\rm B}^{1/2}$ by increasing σ_η over some ranges. In addition, Fig. 3, compared to Fig. 1, uses another set of sampling times $\{t_j\}$, which essentially illustrates that the sampling conditions are not in themselves critical for the existence of the stochastic resonance effect. Usually, only the quantitative details of the effect are influenced by the sampling conditions, but the qualitative feasibility of a nonmonotonic $\bar{\mathcal{E}}_{\rm B}^{1/2}$ is robustly preserved. Fig. 3 also offers numerical validations of the theoretical performance through a Monte Carlo test of the optimal estimator of (3).

For the theoretical evaluations of $\bar{\mathcal{E}}_{\rm B}^{1/2}$ in Figs. 1 and 3, the infinite sums of (12) or (13) have been truncated by considering the zero-mean densities $f_\eta(u)$ to be negligible outside the interval $[-6\sigma_\eta, 6\sigma_\eta]$, which provides a very good approximation. It is possible to have exact evaluations of these sums when $f_\eta(u)$ is defined to be zero outside a bounded support.

We consider passing a noise uniform over $[-1, 1]$ through the nonlinearity

$$g(u) = \frac{1}{a}\frac{\beta u}{\sqrt{1+(\beta u)^2}} \quad (21)$$

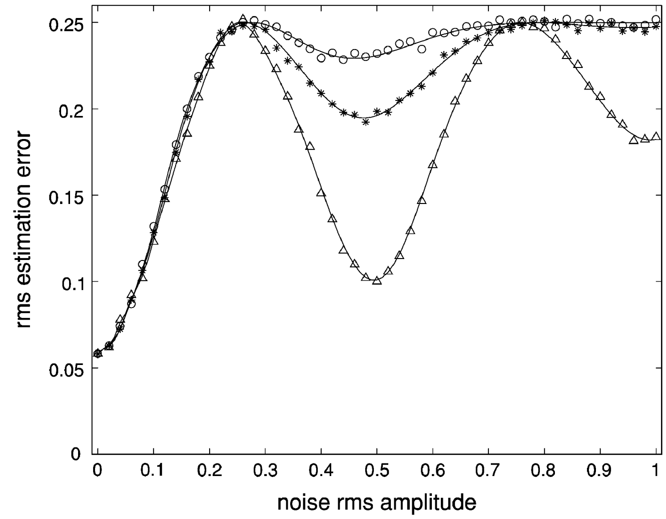


Fig. 3. Rms error $\bar{\mathcal{E}}_{\rm B}^{1/2}$ of the optimal estimator as a function of the rms amplitude σ_η of the Gaussian-mixture noise $\eta(t)$ with density $f_{\rm gm}(u/\sigma_\eta)/\sigma_\eta$ from (19). The solid lines are $\bar{\mathcal{E}}_{\rm B}^{1/2}$ from the theory of (17); the discrete points are $\bar{\mathcal{E}}_{\rm B}^{1/2}$ numerically evaluated from 5×10^3 Monte Carlo trials of the optimal estimator of (3) for each σ_η with $m = 0.9(\circ), m = 0.95(*), m = 0.99(\triangle)$. Prior pdf $p_\nu(u)$ is Gaussian with $m_\nu = 1$ and $\sigma_\nu = 0.25$ and $N = 6$ data samples equispaced with time step 0.2 from $t_1 = 0$ to $t_6 = 1$.

parameterized by $\beta > 0$, with $a = \sqrt{1 - \arctan(\beta)/\beta}$. This produces a standardized noise whose pdf $f_{\rm sq}(u)$ is zero for u outside $[-g(1), g(1)]$, and otherwise

$$f_{\rm sq}(u) = \frac{1}{2\beta}\frac{a}{[1-(au)^2]^{3/2}} \quad (22)$$

and its cumulative distribution function is

$$F_{\rm sq}(u) = \frac{1}{2} + \frac{1}{2\beta}\frac{au}{\sqrt{1-(au)^2}} \quad (23)$$

over the support $u \in [-g(1), g(1)]$, and $F_{\rm sq}(u) = 0$ for $u < -g(1)$ and $F_{\rm sq}(u) = 1$ for $u > g(1)$. As $\beta \to 0$, one recovers the uniform noise over $[-\sqrt{3}, \sqrt{3}]$. For increasing β, the pdf $f_{\rm sq}(u)$ develops "shoulders" about its two modes in $-g(1)$ and $g(1)$, up to $\beta \to +\infty$, which yields a binary noise at ± 1. Some examples of the pdf $f_{\rm sq}(u)$ of (22), for different β, are plotted in Fig. 2(b).

With $f_\eta(u) = f_{\rm sq}(u/\sigma_\eta)/\sigma_\eta$ and $F_\eta(u) = F_{\rm sq}(u/\sigma_\eta)$, we have observed that any $\beta > 0$ can yield nonmonotonic evolutions of the rms error $\bar{\mathcal{E}}_{\rm B}^{1/2}$ of the optimal estimator as σ_η is raised. This is illustrated in Fig. 4 for a Gaussian prior $p_\nu(u)$ and in Fig. 5 for a uniform prior $p_\nu(u)$. The noise reduction of $\bar{\mathcal{E}}_{\rm B}^{1/2}$, as visible in Figs. 4 and 5, gets more pronouced as β grows. At the limit of binary noise ($\beta = +\infty$), Figs. 4 and 5 show that appropriate levels of noise can even reduce $\bar{\mathcal{E}}_{\rm B}^{1/2}$ to its value in the absence of noise.

The conditions of Fig. 5 also reveal a property that is minute in appearance but conceptually significant: Starting from $\sigma_\eta = 0$, the rms error $\bar{\mathcal{E}}_{\rm B}^{1/2}$ first experiences a decaying evolution as σ_η is raised up to $\sigma_\eta \approx 0.01$, where $\bar{\mathcal{E}}_{\rm B}^{1/2}$ starts to rise. This brief decaying excursion altogether represents a relative variation of around 2% in $\bar{\mathcal{E}}_{\rm B}^{1/2}$. The results of Fig. 5 have been obtained through numerical evaluation of the integrals in ν defining $\bar{\mathcal{E}}_{\rm B}^{1/2}$ via (16) and (17). For the integration, the uniform pdf $p_\nu(\nu)$ over

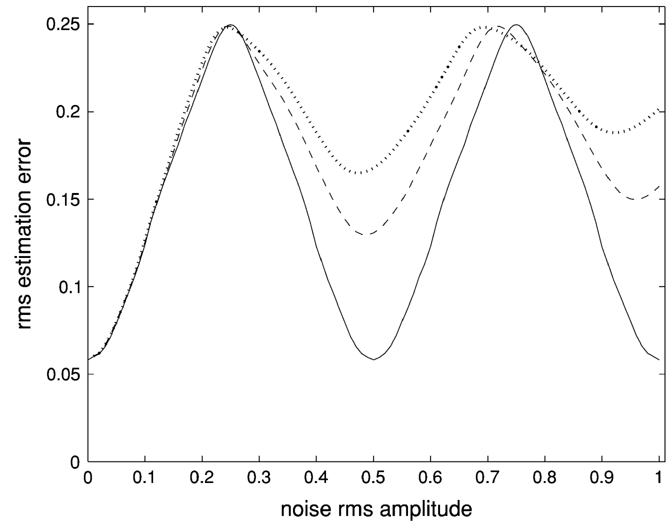


Fig. 4. Rms error $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$ from (17) of the optimal estimator, as a function of the rms amplitude σ_η of the noise $\eta(t)$ distributed according to (23) with $\beta = 5$ (dotted line), $\beta = 10$ (dashed line), and $\beta = +\infty$ (solid line). Prior pdf $p_\nu(u)$ is Gaussian with $m_\nu = 1$ and $\sigma_\nu = 0.25$, and $N = 6$ data samples equispaced with time step 0.2 from $t_1 = 0$ to $t_6 = 1$.

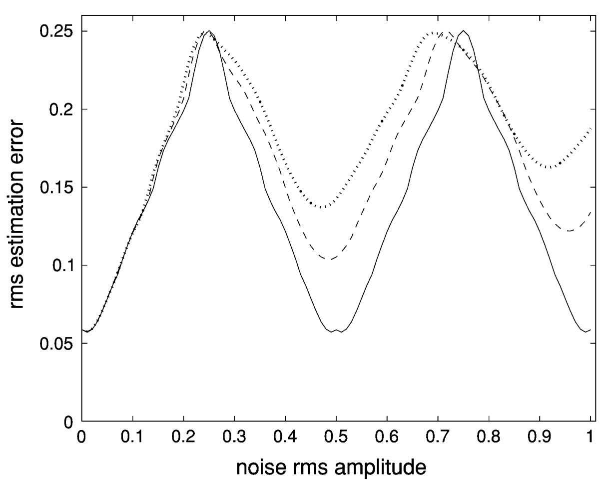


Fig. 5. Same as in Fig. 4, except that $p_\nu(u)$ is uniform over $[m_\nu - \sqrt{3}\sigma_\nu, m_\nu + \sqrt{3}\sigma_\nu]$, with $m_\nu = 1$ and $\sigma_\nu = 0.25$.

its bounded support has been sampled with step $\Delta\nu = 10^{-3}$. For a noise $\eta(t)$ with bounded support and an analytic cumulative distribution $F_\eta(u)$ of (23), the infinite sum of (13) reduces to a finite sum, and the probabilities $\Pr\{x_j \,|\, \nu\}$ are exactly computable. The sampling step for σ_η in Fig. 5 is $\Delta\sigma_\eta = 0.01$. These conditions of the numerical computation in Fig. 5 are just at the limit for discerning the small decaying excursion of $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$ about the origin $\sigma_\eta = 0$, yet we were able to check this behavior with an exact analytical computation. For $p_\nu(\nu)$ uniform and $\eta(t)$ with bounded support, and for σ_η small, we developed up to completion an exact analytical computation of $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$. In the time-sampling conditions of Fig. 5 with $N = 6$ data samples, a number of $2^N = 64$ expressions of the form of (16) have been analytically evaluated and summed up to yield $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$ from (17). The outcome of this exact analytical computation confirms the decaying excursion $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$ about the origin $\sigma_\eta = 0$, as revealed by the numerical computation shown in Fig. 5. This signifies that conditions exist for the optimal estimator where a nonzero optimal level of noise can improve upon the performance in the absence of noise. Such a behavior was known for stochastic resonance in suboptimal devices, where the performance at zero noise is worst and starts to improve as the noise grows, but it is shown for the first time here for an optimal device.

For a prior $p_\nu(\nu)$ with an unbounded support, i.e., the Gaussian case of Figs. 1–4, we were not able to complete a similar exact analytical computation for the behavior of $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$ about the origin $\sigma_\eta = 0$, and the finite-precision numerical computation of these figures shows a (more common) increasing evolution of $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$ as σ_η starts to grow above zero.

V. Estimation on a Sine Wave

The case of a square wave with phase noise that we have considered so far allowed us to implement both a theoretical and a numerical analysis of the optimal Bayesian estimator. This double approach in conjunction was important here for a cross-validation in the demonstration of the feasibility of a noise-enhanced performance of the optimal estimator.

An important case in practice is estimation on a sine wave. The general Bayesian framework of Section II applies equally in this case, but the theoretical analysis cannot be easily worked out in a similar complete fashion. When the observable signal $x(t)$ of (7) is realized with the sine wave $w(t) = \sin(2\pi t)$, the key element that opens the way to the optimal Bayesian estimator is, as before, the pdf $p(x_j \,|\, \nu)$, with its appropriate expression replacing (8). In the case of the sine wave, we have

$$p(x_j \,|\, \nu) dx_j = \Pr\{\sin(2\pi[\nu t_j + \eta(t_j)]) \in [x_j, x_j + dx_j)\}. \tag{24}$$

Keeping track of all the possibilites under which the event on the right-hand side of (24) can take place, in a similar way as in (10)–(13), according to the realizations of the noise $\eta(t_j)$, we finally get

$$\begin{aligned} p(x_j \,|\, \nu) = & \frac{1}{2\pi\sqrt{1 - x_j^2}} \\ & \times \sum_{k=-\infty}^{+\infty} \left\{ f_\eta \left[\frac{1}{2\pi} \arcsin(x_j) - \nu t_j + k \right] \right. \\ & \left. + f_\eta \left[\frac{1}{2} - \frac{1}{2\pi} \arcsin(x_j) - \nu t_j + k \right] \right\}. \end{aligned} \tag{25}$$

As before, through the Bayes rule (5), (25) provides access to the optimal Bayesian estimate of (3), and to its performance measured by $\bar{\mathcal{E}}_{\mathrm{B}}$ of (6), which now comes under the form

$$\bar{\mathcal{E}}_{\mathrm{B}} = \int dx_1 \cdots \int dx_N \mathcal{E}_{\mathrm{B}}(\boldsymbol{x}) \int_{-\infty}^{+\infty} p_\nu(\nu) \prod_{j=1}^{N} p(x_j \,|\, \nu)\, d\nu. \tag{26}$$

In the previous case of the square wave, the theoretical performance $\bar{\mathcal{E}}_{\mathrm{B}}$ of (17) is expressed by a discrete sum over the 2^N states possible for the discrete data $\boldsymbol{x}$. By contrast, in the case of the sine wave, $\bar{\mathcal{E}}_{\mathrm{B}}$ of (26) is expressed by an N-dimensional integral over the continuous data $\boldsymbol{x}$ varying in $\mathbb{R}^N$; in practice, this makes the numerical evaluation of $\bar{\mathcal{E}}_{\mathrm{B}}$ a much heavier task.

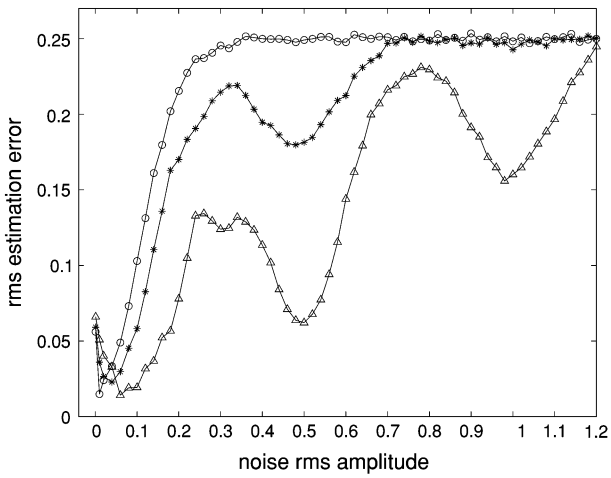


Fig. 6. Rms error $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$ of the optimal estimator as a function of the rms amplitude σ_η of the noise $\eta(t)$. Error $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$ is numerically evaluated from 10^4 Monte Carlo trials of the optimal estimator from (3) and (25) for each σ_η with $\eta(t)$ Gaussian ($\circ$), $\eta(t)$ Gaussian mixture with $m = 0.95(*)$, and $m = 0.99(\triangle)$. Prior pdf $p_\nu(u)$ is Gaussian with $m_\nu = 1$ and $\sigma_\nu = 0.25$, and $N = 6$ data samples equispaced with time step 0.2 from $t_1 = 0$ to $t_6 = 1$. The solid lines here are merely a guide for the eye and not the result of a computation, as opposed to Fig. 3.

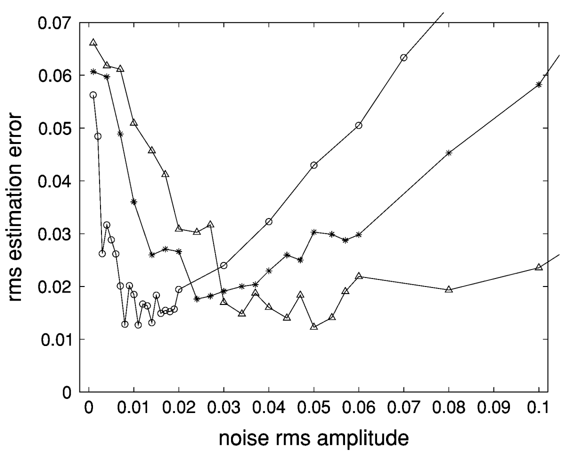


Fig. 7. Same as in Fig. 6 but with a finer resolution over the region around the origin $\sigma_\eta = 0$.

Alternatively, instead of a numerical evaluation of the multiple integral of (26), a Monte Carlo evaluation of $\bar{\mathcal{E}}_{\mathrm{B}}$ can be envisaged, as was previously done in Fig. 3.

Fig. 6 shows results of this Monte Carlo evaluation of the performance of the optimal Bayesian estimator with a sine wave for different types of noise $\eta(t)$. As visible in Fig. 6, the stochastic resonance effect as a nonmonotonic error $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$ is still feasible in this case, in similar conditions as with the square wave, essentially with non-Gaussian noise.

Moreover, the evolutions of Fig. 6 display again the interesting behavior already present in Fig. 5 of a decreasing error $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$ around the origin $\sigma_\eta = 0$. For a better appreciation, Fig. 7 presents other evolutions of $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$ around $\sigma_\eta = 0$ at a finer resolution. In spite of the fluctuations attached to the Monte Carlo evaluations of $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$, clear decreasing trends are visible in Fig. 7 for $\bar{\mathcal{E}}_{\mathrm{B}}^{1/2}$ around the origin. This again points to the possibility, in principle, of a performance for the optimal estimator, which will be better in the presence of a (small) nonzero amount of noise, rather than in the absence of noise. This even becomes possible with Gaussian noise, in the conditions of Fig. 7, extending the possibility of a stochastic resonance (although small here) to Gaussian noise. Further studies will be useful to extend this special aspect of stochastic resonance.

VI. Discussion

Stochastic resonance teaches us that in "nonstandard" conditions of signal-noise coupling, i.e., nonlinear coupling, non-Gaussian noise, the noise is not necessarily a nuisance but may sometimes reveal beneficial through some cooperative interaction with the signal. It has appeared, since its introduction, that such an effect of improvement by noise can occur under many different modalities. These modalities are still largely under inventory and investigation: a necessary stage before discerning whether or how stochastic resonance can be involved in practical techniques for signal processing. Here, we propose a new step in the inventory and exploration of the potentialities of stochastic resonance through the formulation and demonstration of a novel form in optimal estimation.

Standard forms of stochastic resonance usually consider a fixed system, in charge of the processing of a signal, and reveal how noise enhancement can improve the performance of such a fixed system. These fixed systems are usually considered for their own sake, without explicit consideration of their situation relative to the optimal system for the intended processing. Here, we chose to analyze the performance of the optimal system (the optimal Bayesian estimator). It can be pointed out that this optimal system is noise dependent, and in this respect, it differs from the fixed systems usually analyzed in standard stochastic resonance. In this respect, our presentation can be seen as an extension to the standard concept of stochastic resonance. On another level, the interpretation can be that we are considering the same system, *the* optimal Bayesian estimator. The central consideration for us here is that all of these processes represent situations where an increase in the noise level can produce an improvement of the processing. This is the common unifying feature that we see at the root of the concept of stochastic resonance, and that, for us, motivates a uniform treatment.

In the novel form of stochastic resonance we investigate here, several important elements play a part in the effect: the prior distribution $p_\nu(u)$ of the parameter, the type of the noise, the type of the periodic waveform, and the sampling conditions. We have shown, with various illustrative sets of configurations, that the feasibility of the effect does not critically depend on very specific choices for these elements but that it can be robustly preserved over reasonably broad conditions. Beyond this, detailed analyses of the influence of each element, in conjunction with the others, remain open for future work. Such analyses are directly possible, in principle, within the framework we developed here.

Based on the tested configurations, it seems that the effect of improvement by noise gets more pronounced as one departs more and more from Gaussian noise to approach binary noise. This is true at least for our observations with a square wave, and in addition, based on Figs. 6 and 7, the effect is still possible for

Gaussian noise. Further studies will be useful to better appreciate the importance of the non-Gaussian or Gaussian character of the noise for the present form of the effect as well as for stochastic resonance possibly in other optimal processes.

If the purpose is to extract benefit of the reported effect through purposeful addition of noise [for instance, via an additional random motion exerted on the transducer mentioned in the paragraph after (7)], then one needs to be able to increase the level of noise while controlling its nature and especially its pdf. This will be directly feasible with the Gaussian pdf of Figs. 6 and 7, whose form remains unchanged if more Gaussian noise is added. In other non-Gaussian cases, the control of the pdf while more noise is added is a more complex issue that is not explicitly addressed here. If the pdf of the noise changes as its rms amplitude increases, the analysis we worked out is not sufficient and has to be complemented by an explicit description of the way the pdf changes as more noise is added. Yet, since our results show that a stochastic resonance effect can be preserved over broad classes of different noise pdf, an improvement may still be possible when the noise pdf changes while its rms value increases. In addition, a more internal adjustable parameter, playing a role similar to a physical temperature, may be available, depending on the context, to increase the level of noise while maintaining its pdf. The elucidation of such issues will require further studies and, maybe, evolutions in the setting and conditions considered here that will complement our knowledge of stochastic resonance. Particularly, beyond the randomly moving transducer example evoked above, the exploration of other settings where it is possible and useful to control and add phase noise to a signal constitutes a perspective for future study.

The main focus of the present work is to demonstrate that in principle, some form of improvement by noise, which characterizes stochastic resonance, is not restricted to suboptimal processing but may also apply to optimal processing. Similar noise enhancement may also exist in other types of operations. The demonstration of stochastic resonance in optimal processing is obtained here for estimation of a random parameter in a Bayesian framework. Distinct approaches to estimation, for instance estimation of a nonrandom parameter in a maximum likelihood framework, could also be considered in the same perspective. Such approaches are based on a different problem formulation, and they seek to optimize a different measure of performance. In essence, therefore, they are not directly comparable to the present Bayesian approach. Yet, a meaningful question is whether a maximum likelihood estimation of a nonrandom parameter could also lend itself to a stochastic resonance effect under the form of a noise-improved performance. This issue remains open for investigation. In the same perspective, studies are currently under way to investigate the possibility of extending stochastic resonance to optimal detection [36].

The possibility of noise-improved processing may find applicability in complex environments with nonlinear or non-Gaussian conditions, for instance, in multisensor intelligent systems involved in real-time processing. Neural systems are natural systems of this kind. They strongly rely on nonlinear processing in noisy environments of signals made of pulses that are invariant in shape coding information through their phase or timing, and stochastic resonance is an available property shown in these systems. In such complex nonlinear situations, stochastic resonance may play a part in maintaining high performance for signal processing. The novel form of stochastic resonance we have demonstrated here, together with further developments, will contribute to this perspective, which aims to improve nonlinear processing.

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