14th International Conference on Mathematical Theory of Networks and Systems
Perpignan, France, 19–23 June 2000 ; pp. 79–82.

Information-theoretic measures improved by noise in nonlinear systems

François CHAPEAU-BLONDEAU, Julio ROJAS-VARELA
Laboratoire d'Ingénierie des Systèmes Automatisés (LISA), Université d'Angers,
62 avenue Notre Dame du Lac, 49000 ANGERS, FRANCE.
{chapeau,rojas}@univ-angers.fr, http://www.istia.univ-angers.fr/~chapeau/

Keywords: Stochastic resonance, nonlinear systems, information theory, signal, noise.

Abstract

Stochastic resonance is a phenomenon whereby the transmission of a signal by certain nonlinear systems can be improved by addition of noise. We propose a brief overview of this effect, together with an extension based on information-theoretic concepts. We analyze various conditions of nonlinear transmission where the input–output Shannon mutual information, the input–output Kullback divergence, or the input–output Fisher information can receive improvement from noise addition, demonstrating different forms of noise-enhanced transmission.

1 Stochastic resonance phenomenon

When a linear system couples linearly a signal and a noise, generally the noise acts as a nuisance spoiling the signal. By contrast, when certain types of nonlinear systems couple nonlinearly a signal and a noise, there may exist cooperative conditions where the noise benefits to the signal, up to a point where adding noise may improve the transmission of the signal by the nonlinear system. This (counterintuitive) phenomenon, where the efficacy of a nonlinear system in transmitting a signal may be improved by noise, is known under the name of stochastic resonance [1, 2]. Since its introduction some twenty years ago [3], stochastic resonance has been reported in nonlinear systems pertaining to a broad variety of domains, including electronics [4, 5, 6], mechanics [7], optics [8, 9], neurobiology [10, 11, 12].

Very often, stochastic resonance can be cast under the general scheme that follows. A "coherent" or information-carrying signal $s(t)$ added to a noise $\eta(t)$ are input onto a nonlinear transmission system which, in response, produces the output signal $y(t)$. In general, because of the influence of the random input $\eta(t)$, the output $y(t)$ is a random signal, but which also bears some dependence on the coherent input $s(t)$. A measure to quantify this dependence is then specified, according to the nature of the signals and of the transmission system. Stochastic resonance then consists in the possibility of improving this measure of dependence of $y(t)$ on $s(t)$ by means of an increase in the level of the noise $\eta(t)$.

Most of the time, stochastic resonance has been exhibited for a deterministic signal of known form $s(t)$, essentially a periodic signal, whose transmission by various nonlinear systems was shown improvable via noise addition [13, 1]. In the case of a periodic signal, the measure of the transmission efficacy receiving improvement from the noise, is a signal-to-noise ratio evaluated in the frequency domain at the output of the transmission system, as the ratio of the power contained at the frequency of the periodic signal divided by the power contributed by the noise. Certain nonlinear systems can then take incoherent energy from the noise and feed it into coherent energy at the frequency of the periodic signal, leading to a possibility of increasing the signal-to-noise ratio at the output by inputting noise via $\eta(t)$ into the system [14, 13]. This way of measuring a noise-improved transmission is possible because the energy of the periodic signal has a well defined frequency localization.

Stochastic resonance though, has recently been extended to the transmission of aperiodic signals $s(t)$ [15]. In this case, correlation measures between the output $y(t)$ and the aperiodic coherent input $s(t)$ have been proposed to quantify the efficacy of the nonlinear transmission. Nonlinear systems have then been exhibited where this input–output correlation can be enhanced by increasing the noise $\eta(t)$. This characterization of stochastic resonance has been especially applied with aperiodic deterministic signals $s(t)$ of known form, to establish a noise-enhanced transmission by certain nonlinear systems (for instance, neuronal systems) [15].

Even more recently, stochastic resonance has been extended for the noise-improved transmission of random information-carrying signals $s(t)$. In this case, correlation measures can also be used for the transmission efficacy, yet particularly appropriate measures are provided by information-theoretic quantities [16, 17, 18]. Nonlinear systems have been exhibited where the mutual information between the output $y(t)$ and the information-carrying input $s(t)$ can be increased via injection of noise into the system. Here we shall add other types of information-theoretic measures for the characterization of stochastic resonance in nonlinear transmission. In addition to the mutual information, we shall show that the Kullback divergence and the Fisher information can provide meaningful measures for characterizing forms of stochastic resonance. The application of the Kullback divergence is new to stochastic resonance. The first applications of the Fisher information to stochastic resonance have appeared in [19] for signal transmission by neurons, then in [20] via the related Cramér-Rao bound. Here, it will be applied to a different nonlinear system, in some sense conceptually simpler, and within the unified view of several information-theoretic characterizations explicitly evaluated for the same nonlinearity.

In the nonlinear context of stochastic resonance, it is not always the case that a theoretical (analytical) demonstration of the effect can be obtained, and many studies on stochastic resonance have relied on numerical simulation or experiment to establish the effect. By contrast here, we shall report stochastic resonance with a nonlinear system simple enough to lend itself to an exact theoretical treatment. In various conditions of nonlinear signal transmission in the presence of noise, we shall explicitly compute a Shannon mutual information $I(s;y)$ between $y(t)$ and $s(t)$, a Kullback divergence $D(s;y)$ between the probability laws of $y(t)$ and $s(t)$, and a Fisher information $J(s;y)$ between $y(t)$ and $s(t)$. We shall exhibit conditions where all three information-theoretic quantities can be improved via noise addition, revealing different forms of noise-enhanced signal transmission.

2 A nonlinear transmission system and its information-theoretic characterization

In order to have demonstrations of the effect analytically tractable with these information-theoretic measures, we consider the simple situation that follows. The transmission is described as $y(t) = \Gamma[s(t) + \eta(t) - \theta]$ with the step nonlinearity $\Gamma(u) = 1$ if $u > 0$ and $\Gamma(u) = 0$ otherwise, and a threshold θ. The signals $s(t)$, $\eta(t)$ and $y(t)$ are observed or sampled at discrete times t_k. We suppose that the signal s at the sampling times t_k assumes, just as y, values restricted to 1 or 0, respectively with probabilities $\Pr\{s = 1\} = p_1$ and $\Pr\{s = 0\} = 1 - p_1$. The transmission system can then be interpreted as a binary information channel. The input–output Kullback divergence $D(s;y)$ [21] between the probability distributions of y and s is defined as

$$D(s;y) = \sum_{x=0,1} \Pr\{s = x\} \log_2 \left(\frac{\Pr\{s = x\}}{\Pr\{y = x\}} \right) . \quad (1)$$

It can be explicitly expressed as

$$D(s;y) = p_1 \log_2 \left(\frac{p_1}{q_1} \right) + (1 - p_1) \log_2 \left(\frac{1 - p_1}{1 - q_1} \right) , \quad (2)$$

with $q_1 = \Pr\{y = 1\} = p_1 p_{11} + (1 - p_1) p_{10}$ and

$$p_{11} = \Pr\{y = 1 \,|\, s = 1\} = 1 - F_\eta(\theta - 1) , \quad (3)$$

$$p_{10} = \Pr\{y = 1 \,|\, s = 0\} = 1 - F_\eta(\theta) , \quad (4)$$

F_η being the cumulative distribution function of the noise $\eta(t)$.

For illustration, Fig. 1 represents an evolution of $D(s;y)$ from Eq. (2), as a function of the rms amplitude σ_η of the noise η. Figure 1 shows a region where $D(s;y)$ decreases as σ_η increases, meaning that adding noise may make the random signals $s(t)$ and $y(t)$ more similar, up to an optimal nonzero noise level where the input–output similarity is maximized.

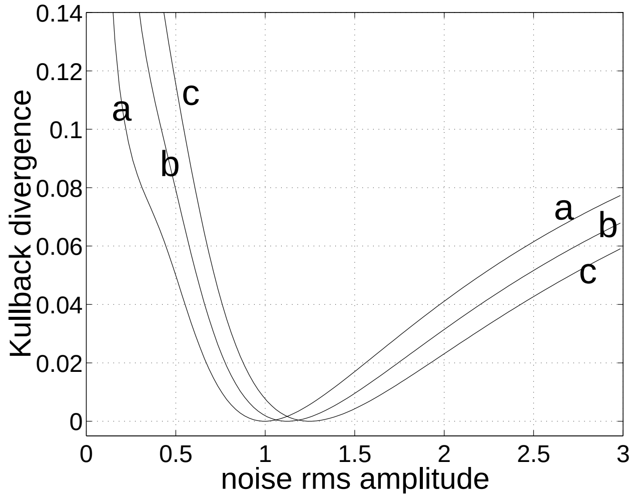

Figure 1: Input–output Kullback divergence $D(s;y)$ from Eq. (2), as a function of the rms amplitude σ_η of the noise η chosen zero-mean Gaussian, when $p_1 = 0.2$ and $\theta = 1.1$ (a), $\theta = 1.2$ (b), $\theta = 1.3$ (c).

If we further assume that both $s(t)$ and $\eta(t)$ are white random signals, the transmission system can then be interpreted as a *memoryless* binary information channel. Its input–output Shannon mutual information $I(s;y)$ [21] can be defined as

$$I(s;y) = H(y) - H(y|s) , \quad (5)$$

with the output entropy

$$H(y) = \sum_{x=0,1} h(\Pr\{y = x\}) \quad (6)$$

and the input–output conditional entropy

$$\begin{aligned} H(y|s) &= p_1 \sum_{x=0,1} h(\Pr\{y = x \,|\, s = 1\}) + \\ &\quad (1 - p_1) \sum_{x=0,1} h(\Pr\{y = x \,|\, s = 0\}) , \end{aligned} \quad (7)$$

where $h(u) = -u \log_2(u)$. These entropies can be explicitly evaluated as

$$\begin{aligned} H(y) &= h[p_1 p_{11} + (1 - p_1) p_{10}] + \\ &\quad h[p_1(1 - p_{11}) + (1 - p_1)(1 - p_{10})] \end{aligned} \quad (8)$$

and

$$\begin{aligned} H(y|s) &= p_1 [h(p_{11}) + h(1 - p_{11})] + \\ &\quad (1 - p_1)[h(p_{10}) + h(1 - p_{10})] , \end{aligned} \quad (9)$$

whence an explicit expression for the input–output mutual information $I(s;y)$ of Eq. (5).

Again for illustration, Fig. 2 represents an evolution of $I(s;y)$ from Eq. (5), as a function of the rms amplitude σ_η of the noise η. Figure 2 shows a region where $I(s;y)$ increases as σ_η is raised, characterizing an effect of noise-assisted information transmission. The effect takes place when the signal $s(t)$ by itself is too small to overcome the threshold; in the absence of noise no information is transmitted, as expressed by a vanishing $I(s;y)$ at zero noise in Fig. 2. Addition of noise then brings assistance to the signal in overcoming the threshold; input–output transmission of information can thus occur, as expressed by the rise of $I(s;y)$ which culminates at a maximum input–output transmission for an optimal nonzero noise level.

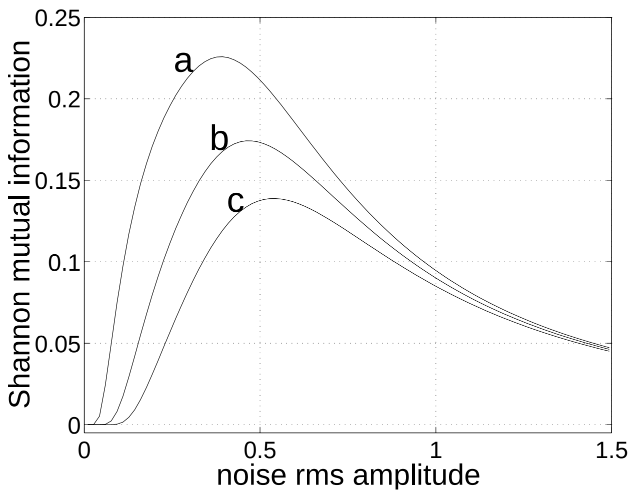

Figure 2: Input–output Shannon mutual information $I(s;y)$ from Eq. (5), as a function of the rms amplitude σ_η of the noise η chosen zero-mean Gaussian, when $p_1 = 0.5$ and $\theta = 1.1$ (a), $\theta = 1.2$ (b), $\theta = 1.3$ (c).

We now move to a different hypothesis concerning the information-carrying signal $s(t)$. We assume that the information to be recovered from the observed output signal $y(t)$, is the value of some parameter characterizing the signal $s(t)$ buried in the noise $\eta(t)$. For a simple illustration we assume that $s(t)$ reduces to a constant value s, and we seek to estimate s from observations of the signal $y(t) = \Gamma[s+\eta(t)-\theta]$. The Fisher information $J(s;y)$ [21] contained in $y(t)$ about s, which limits the performance of all unbiased estimators of s from y, can be defined as

$$J(s;y) = \sum_{x=0,1} \frac{1}{\Pr\{y=x\}} \left(\frac{\partial}{\partial s} \Pr\{y=x\} \right)^2 . \qquad (10)$$

It can be explicitly evaluated under the form

$$J(s;y) = \left(\frac{1}{q_1} + \frac{1}{1-q_1} \right) \left(\frac{\partial q_1}{\partial s} \right)^2 \qquad (11)$$

where again $q_1 = \Pr\{y=1\}$. We also have $q_1 = \Pr\{s+\eta > \theta\}$, which here with s fixed gives $q_1 = \Pr\{\eta > \theta - s\}$, amouting to $q_1 = 1 - F_\eta(\theta - s)$. For the derivative we thus have $\partial q_1/\partial s = f_\eta(\theta - s)$, with f_η the probability density function of the noise $\eta(t)$.

Fisher information $J(s;y)$ of Eq. (11) then results as

$$J(s;y) = \frac{f_\eta^2(\theta - s)}{F_\eta(\theta - s)[1 - F_\eta(\theta - s)]} . \qquad (12)$$

Figure 3 represents an illustrative evolution of $J(s;y)$ from Eq. (12), as a function of the rms amplitude σ_η of the noise η. Figure 3 shows a region where $J(s;y)$ increases as σ_η is raised, characterizing a larger possible efficacy in the estimation when noise is added, up to an optimal nonzero noise level where the estimation efficacy is maximized.

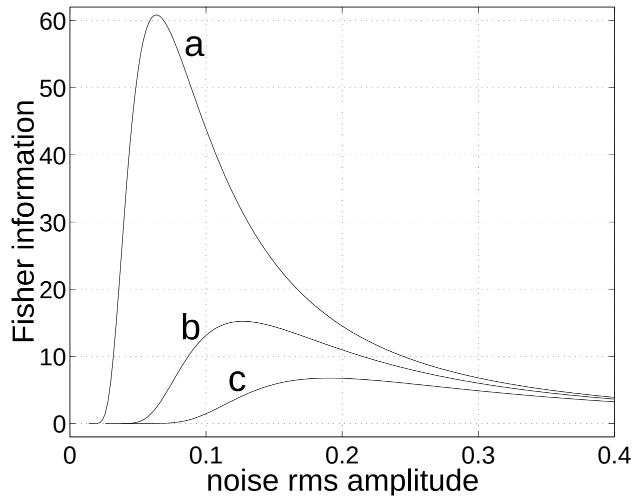

Figure 3: Input–output Fisher information $J(s;y)$ from Eq. (12), as a function of the rms amplitude σ_η of the noise η chosen zero-mean Gaussian, when $\theta = 1$ and $s = 0.9$ or 1.1 (a), $s = 0.8$ or 1.2 (b), $s = 0.7$ or 1.3 (c).

3 Conclusion

The conditions we have presented, of various forms of noise-enhanced transmission with information-theoretic characterizations, are merely illustrative. The effect is preserved over a broad range of signals and nonlinear systems, as it was verified to be the case in studies on other forms of stochastic resonance. Also different measures can be used to quantify a stochastic resonance effect, especially measures from information theory as we showed here, all depending on the purpose and prospect involved.

More generally, stochastic resonance, of relatively recent introduction, remains an emerging effect. From a conceptual standpoint, stochastic resonance is an important phenomenon as it modifies the status of the noise by establishing that in nonlinear systems noise is not necessarily a nuisance but may sometimes be turned into a benefit. From a practical standpoint, stochastic resonance may have useful applications for signal processing by nonlinear systems, especially when no full control is available over the nonlinearities. Both aspects of stochastic resonance call for further exploration.

References

[1] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, "Stochastic resonance," *Reviews of Modern Physics*, vol. 70, pp. 223–287, 1998.

[2] S. Mitaim and B. Kosko, "Adaptive stochastic resonance," *Proceedings of the IEEE*, vol. 86, pp. 2152–2183, 1998.

[3] R. Benzi, A. Sutera, and A. Vulpiani, "The mechanism of stochastic resonance," *Journal of Physics A*, vol. 14, pp. L453–L458, 1981.

[4] V. S. Anishchenko, M. A. Safonova, and L. O. Chua, "Stochastic resonance in the nonautonomous Chua's circuit," *Journal of Circuits, Systems and Computers*, vol. 3, pp. 553–578, 1993.

[5] X. Godivier and F. Chapeau-Blondeau, "Noise-assisted signal transmission in a nonlinear electronic comparator: Experiment and theory," *Signal Processing*, vol. 56, pp. 293–303, 1997.

[6] X. Godivier, J. Rojas-Varela, and F. Chapeau-Blondeau, "Noise-assisted signal transmission via stochastic resonance in a diode nonlinearity," *Electronics Letters*, vol. 33, pp. 1666–1668, 1997.

[7] M. Spano, M. Wun-Fogle, and W. L. Ditto, "Experimental observation of stochastic resonance in a magnetoelastic ribbon," *Physical Review A*, vol. 46, pp. 5253–5256, 1992.

[8] B. M. Jost and B. E. A. Saleh, "Signal-to-noise ratio improvement by stochastic resonance in a unidirectional photorefractive ring resonator," *Optics Letters*, vol. 21, pp. 287–289, 1996.

[9] F. Vaudelle, J. Gazengel, G. Rivoire, X. Godivier, and F. Chapeau-Blondeau, "Stochastic resonance and noise-enhanced transmission of spatial signals in optics: The case of scattering," *Journal of the Optical Society of America B*, vol. 15, pp. 2674–2680, 1998.

[10] A. Bulsara, E. W. Jacobs, T. Zhou, F. Moss, and L. Kiss, "Stochastic resonance in a single neuron model: Theory and analog simulation," *Journal of Theoretical Biology*, vol. 152, pp. 531–555, 1991.

[11] F. Chapeau-Blondeau and X. Godivier, "Stochastic resonance in nonlinear transmission of spike signals: An exact model and an application to the neuron," *International Journal of Bifurcation and Chaos*, vol. 6, pp. 2069–2076, 1996.

[12] X. Godivier and F. Chapeau-Blondeau, "Noise-enhanced transmission of spike trains in the neuron," *Europhysics Letters*, vol. 35, pp. 473–477, 1996.

[13] F. Chapeau-Blondeau and X. Godivier, "Theory of stochastic resonance in signal transmission by static nonlinear systems," *Physical Review E*, vol. 55, pp. 1478–1495, 1997.

[14] K. Wiesenfeld and F. Moss, "Stochastic resonance and the benefits of noise: From ice ages to crayfish and SQUIDs," *Nature*, vol. 373, pp. 33–36, 1995.

[15] J. J. Collins, C. C. Chow, and T. T. Imhoff, "Aperiodic stochastic resonance in excitable systems," *Physical Review E*, vol. 52, pp. R3321–R3324, 1995.

[16] C. Heneghan, C. C. Chow, J. J. Collins, T. T. Imhoff, S. B. Lowen, and M. C. Teich, "Information measures quantifying aperiodic stochastic resonance," *Physical Review E*, vol. 54, pp. R2228–R2231, 1996.

[17] F. Chapeau-Blondeau, "Noise-enhanced capacity via stochastic resonance in an asymmetric binary channel," *Physical Review E*, vol. 55, pp. 2016–2019, 1997.

[18] X. Godivier and F. Chapeau-Blondeau, "Stochastic resonance in the information capacity of a nonlinear dynamic system," *International Journal of Bifurcation and Chaos*, vol. 8, pp. 581–590, 1998.

[19] M. Stemmler, "A single spike suffices: The simplest form of stochastic resonance in model neurons," *Network: Computation in Neural Systems*, vol. 7, pp. 687–716, 1996.

[20] H. C. Papadopoulos, G. W. Wornell, and A. V. Oppenheim, "Low-complexity digital encoding strategies for wireless sensors networks," in *Proceedings IEEE International Conference on Acoustics, Speech and Signal Processing*, pp. 3273–3276, 1998.

[21] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.