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# OPTICAL COHERENCE OF A SCALAR WAVE INFLUENCED BY FIRST-ORDER AND SECOND-ORDER STATISTICS OF ITS RANDOM PHASE

FRANÇOIS CHAPEAU-BLONDEAU[1], DENIS GINDRE[2], RÉGIS BARILLÉ[2], DAVID ROUSSEAU[1]

[1] *Laboratoire d'Ingénierie des Systèmes Automatisés (LISA), Université d'Angers, 62 avenue Notre Dame du Lac, 49000 Angers, France*

[2] *Laboratoire des Propriétés Optiques des Matériaux et Applications (POMA), Université d'Angers, 2 boulevard Lavoisier, 49000 Angers, France*

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We analyze a simple model of a scalar optical wave with partial coherence. The model is devised to describe the influence on the coherence of the wave, of the statistical properties of its random phase, including both the second-order statistics (phase correlation) — which is classic, but also the first-order statistics (phase distribution) — which is nonclassic. Expectedly, upon increasing the disorder of the fluctuating phase through a reduction of its correlation duration, the model shows that the coherence of the wave is always reduced. By contrast, upon increasing the disorder of the fluctuating phase through an increase of its dispersion, the model reveals that the coherence of the wave can sometimes be enhanced. This beneficial consequence of an increase in disorder is related to the phenomenon of stochastic resonance or improvement by noise in signal processing.

*Keywords*: Optical coherence; phase fluctuations; statistical optics; interferometry; stochastic resonance.

## 1. Introduction

Optical coherence is a macroscopically measurable manifestation of some order in a light wave, which is under the dependence of microscopic degrees of freedom usually nondirectly observable. Optical coherence is a rich notion, which is still experiencing new interesting developments, at the conceptual level, and also thanks to advances in technologies of light sources and optical devices allowing more and more control on light properties [1–5].

Optical coherence is essentially governed by the statistical properties of light fluctuations at the microscopic level [6–8]. A macroscopically observable light usually results from the superposition of elementary radiations emanating from a very large number of microscopic radiators. The ability of radiating in a coordinated way, on average, for a single radiator over time, and for two spatially distinct radiators, relates respectively to temporal and spatial coherence. An important factor is then the correlation structure of the random fluctuations in the phase of the radiated light, at two distinct times or at two distinct points in space, and therefore refers to the second-order statistical properties of the random phase. Accordingly, the second-order statistics of the random phase are at the root of the traditional description of optical coherence. However, for the averages over the microscopic fluctuations that control the coherence, it is plausible that the first-order (at one time or one point in space) statistical properties of the random phase may also play a part. It is this idea which is theoretically explored in this paper.

We analyze a simple model for a light wave which explicitly takes into account both the first- and second-order statistics of the random phase for their impact on the coherence measured by the fringe visibility in an interference experiment. The model captures the conventional behavior of the coherence as a function of the statistical correlation duration of the phase. It also reveals a possible influence on the coherence of the statistical dispersion of the phase. Interestingly, we shall demonstrate here that an increase in the disorder of the phase due to an increase of its statistical dispersion, can sometimes translate into an improvement of the coherence. This type of order-from-disorder phenomenon can be interpreted as a form of stochastic resonance or improvement by noise in signal processing. Stochastic resonance designates situations where an increase in the level of noise is capable of producing an improvement of some measure of performance associated to definite signal or information processing tasks [9–11]. Stochastic resonance has been observed in various processes, including electronic circuits and neuronal systems; examples of stochastic resonance in optics are given in [12–20]. The possibility of improving the coherence of a light wave through an increase in the disorder of its randomly fluctuating phase, as reported here, could represent a new direction to investigate novel forms of stochastic resonance.

## 2. Measuring the Coherence

In this Section 2, we briefly recall some basic notions which will be useful to us as reference for the optical coherence [7, 8], and which we will extend in the following Section 3. A very common model for a scalar wave of angular frequency $\omega_0$ and fixed amplitude $A$, is [6, 8]

$$u(t) = A\cos(\omega_0 t + \phi)\,. \tag{1}$$

When the phase $\phi$ is constant in time, the wave $u(t)$ has complete coherence, it is strictly periodic with period $T_0 = 2\pi/\omega_0$, and a time average over one period defines its intensity

$$I_0 = \frac{1}{T_0}\int_0^{T_0} u^2(t)dt = \frac{A^2}{2}\,. \tag{2}$$

A detector measuring light intensity will usually have an integration time much longer than the period $T_0$, and will rather be sensitive to the "experimental" average

$$\lim_{T_{\rm int}\to\infty} \frac{1}{T_{\rm int}}\int_0^{T_{\rm int}} u^2(t)dt\,. \tag{3}$$

Yet, for a strictly periodic wave $u(t)$ and when $T_{\rm int} \gg T_0$, the experimental average of Eq. (3) precisely matches the one-period average of Eq. (2) and the detector effectively measures the intensity value $I_0 = A^2/2$.

The coherent wave $u(t)$ lends itself to interference phenomena. With an interferometric device, $u(t)$ is split into two waves shifted in time by a pathlength difference $T$, and then the two waves are recombined to form on a screen or a detector the superposition

$$u_{\rm tot}(t) = u(t) + u(t+T) = A\cos(\omega_0 t + \phi) + A\cos(\omega_0 t + \omega_0 T + \phi)\,. \qquad (4)$$

The intensity $I$ of wave $u_{\rm tot}(t)$, through a time average of $u_{\rm tot}^2(t)$ equivalently via Eq. (2) or Eq. (3), comes out as

$$I = 2I_0[1 + \cos(\omega_0 T)]\,. \qquad (5)$$

Depending on the pathlength difference $T$, the intensity $I$ of Eq. (5) varies between the minimum $I_{\rm min} = 0$ (fully destructive interference) and the maximum $I_{\rm max} = 4I_0$ (fully constructive interference). Accordingly, the detector when moved along the screen measures an interferogram formed by a succession of dark and bright fringes.

Following Michelson, for a interferogram in the vicinity of any pathlength difference $T$, the fringe visibility is defined as [6, 8]

$$\mathcal{V}(T) = \frac{I_{\rm max} - I_{\rm min}}{I_{\rm max} + I_{\rm min}} \qquad (6)$$

where $I_{\rm max}$ and $I_{\rm min}$ are the intensities at the maximum and minimum of the fringe.

Therefore, for a purely coherent wave according to Eq. (1) associated to the intensity pattern of Eq. (5), the resulting visibility comes as

$$\mathcal{V}(T) = 1 \qquad \text{(complete coherence)}\,, \qquad (7)$$

a unity constant throughout the interferogram, which will be our reference in the sequel for a state of complete coherence.

At another extreme, another reference behavior is the case of completely incoherent waves, that would give on the detector the intensity $I = I_0 + I_0 = 2I_0$ associated to

$$\mathcal{V}(T) = 0 \qquad \text{(complete incoherence)}\,, \qquad (8)$$

for any pathlength difference $T$.

At the root, the phase $\phi$ introduced in Eq. (1) is a microscopic parameter usually not directly accessible to measurement. It is the behavior of the intensity pattern in an interference experiment, as summarized by the visibility $\mathcal{V}$ of Eq. (6), which is a measurable manifestation of the state of coherence of the underlying wave, and that we study in the sequel as our measure of coherence.

## 3. Partially Coherent Wave

### 3.1. *Model of the wave and its intensity*

We now introduce a model for a partially coherent scalar wave, which is able to encompass all the intermediate states between complete coherence ($\mathcal{V} = 1$) and complete incoherence ($\mathcal{V} = 0$). In addition, the model will allow us to explicitly describe the effect on the coherence, of the second-order statistics (correlation) of the phase — which is classic, but also of the first-order statistics of the phase — which is nonclassic. Starting from Eq. (1) and following [8], a partially coherent wave can be modeled with a phase $\phi(t)$ varying in time:

$$u(t) = A\cos(\omega_0 t + \phi(t)) . \tag{9}$$

Now we choose to model the evolution of the phase $\phi(t)$ through an empirical or phenomenological description of independent elementary radiators producing random uncoordinated emission. Our modeling choice is specially devised to allow in the model, direct control as a free parameter of the probability density of the random phase, which can be set arbitrarily and differ from a uniform density over $[-\pi, \pi]$. Nevertheless, we will check a posteriori that our empirical modeling assumption for the phase at a microscopic elementary level, leads to a model able to predict, in the common case of a phase uniform over $[-\pi, \pi]$, common macroscopic properties of partially coherent light as they are experimentally observable. So our modeling choice here is to model the phase $\phi(t)$ by a piecewise-constant random signal defined as follows. At random times $t_i$ distributed according to a Poisson process of parameter $1/\tau_c$, signal $\phi(t)$ changes for a new constant value $\phi(t) = \varphi$, the possible values $\varphi$ accessible to $\phi(t)$ being selected independently at random with the probability density function $p_\phi(\varphi)$, as depicted in Fig. 1. The probability density $p_\phi(\varphi)$ has mean $m_\phi$ and standard deviation $\sigma_\phi$. The common choice is to take $p_\phi(\varphi)$ uniform over $[-\pi, \pi]$, but here precisely we want to study the impact of a general, non necessarily uniform, $p_\phi(\varphi)$.

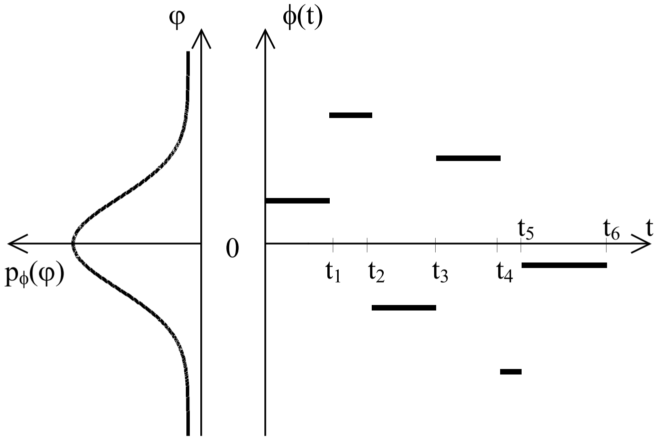


Fig. 1. Random signal $\phi(t)$ modeling the phase, and its probability density function $p_\phi(\varphi)$.

It is possible to define an autocorrelation function for $R_{\phi\phi}(\tau)$ for the random phase $\phi(t)$, whose physical meaning is given by the time average of Eq. (3):

$$R_{\phi\phi}(\tau) = \lim_{T_{\text{int}}\to\infty} \frac{1}{T_{\text{int}}} \int_0^{T_{\text{int}}} \phi(t)\phi(t+\tau)dt \;=\; \overline{\phi(t)\phi(t+\tau)} . \tag{10}$$

The "physical" mean of Eq. (10) is obtained as a time average over one realization $\phi(t)$ of the random phase. Under the assumption of ergodicity of $\phi(t)$, any realization will yield the same mean in Eq. (10). Alternatively, $R_{\phi\phi}(\tau)$ of Eq. (10) can also be computed via statistical or ensemble average, taking care of the doubly stochastic nature of $\phi(t)$ imposed by the temporal Poisson process in abscissa and the amplitude density $p_\phi(\cdot)$ in ordinate. At fixed $t$ and $\tau$, one can consider first the statistical average according to the density $p_\phi(\cdot)$, denoted $\mathrm{E}[\cdot]$. Then, this average $\mathrm{E}[\cdot]$ according to $p_\phi(\cdot)$ is complemented by the average according to the temporal Poisson process, which is denoted $\langle\cdot\rangle$. In principle, the double statistical average $\langle \mathrm{E}[\phi(t)\phi(t+\tau)]\rangle$ matches the time average $\overline{\phi(t)\phi(t+\tau)}$ of Eq. (10).

At fixed $t$ and $\tau \geq 0$, there are two possible configurations for the product $\phi(t)\phi(t+\tau)$:

(i) $\phi(t)\phi(t+\tau) = \varphi_1\varphi_1$, with $\varphi_1$ a random number distributed according to the density $p_\phi(\cdot)$. Then, the statistical average according to $p_\phi(\cdot)$ gives $\mathrm{E}[\varphi_1^2] = m_\phi^2 + \sigma_\phi^2$.
(ii) $\phi(t)\phi(t+\tau) = \varphi_1\varphi_2$, with $\varphi_1$ and $\varphi_2$ two independent random numbers distributed according to the density $p_\phi(\cdot)$. Then, the statistical average according to $p_\phi(\cdot)$ gives $\mathrm{E}[\varphi_1\varphi_2] = m_\phi^2$.

Now configuration (i) holds if no transition of the Poisson process has taken place in interval $[t, t+\tau]$, this outcome occurring with probability $\exp(-\tau/\tau_c)$ according to the properties of the Poisson process. And configuration (ii) holds in the complementary case where at least one transition of the Poisson process has taken place in interval $[t, t+\tau]$, this outcome occurring with probability $1 - \exp(-\tau/\tau_c)$.

The double statistical average of $\phi(t)\phi(t+\tau)$ then gives

$$\langle \mathrm{E}[\phi(t)\phi(t+\tau)]\rangle = [m_\phi^2 + \sigma_\phi^2]\exp(-\tau/\tau_c) + m_\phi^2[1 - \exp(-\tau/\tau_c)]\,, \tag{11}$$

which in principle matches $R_{\phi\phi}(\tau)$ of Eq. (10), leading for any $\tau$ to

$$R_{\phi\phi}(\tau) = m_\phi^2 + \sigma_\phi^2 \exp(-|\tau|/\tau_c)\,, \tag{12}$$

as represented by Fig. 2.

The derivations of Eqs. (10)–(12) show how to relate a physical mean as realized by a detector, to statistical means mathematically computable from the statistical properties of random signals.

We now turn to the intensity $I_0$ which can be attributed to the partially coherent wave $u(t)$ of Eq. (9). This intensity can no longer be defined through a time average of $u^2(t)$ over one period $T_0$ according to Eq. (2), because strictly speaking $u(t)$ of Eq. (9) is no longer periodic, due to the time-varying phase $\phi(t)$. A definition with a well defined meaning is again through the physical mean of Eq. (3). The intensity $I_0$ measured by the detector for the wave $u(t)$ of Eq. (9) is

$$I_0 = \lim_{T_{\mathrm{int}}\to\infty} \frac{1}{T_{\mathrm{int}}} \int_0^{T_{\mathrm{int}}} u^2(t)dt = \overline{u^2(t)}\,. \tag{13}$$

One then has

$$I_0 = A^2\, \overline{\cos^2\big(\omega_0 t + \phi(t)\big)}\;. \tag{14}$$

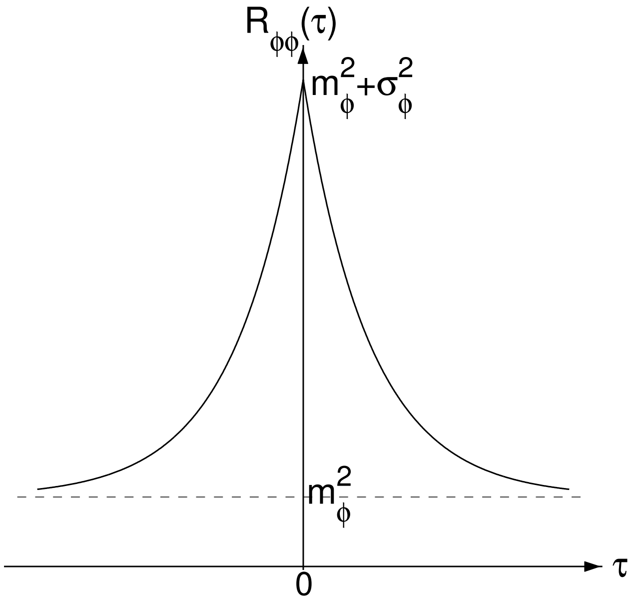


Fig. 2. Autocorrelation function $R_{\phi\phi}(\tau)$ from Eq. (12) for the random phase $\phi(t)$ of Fig. 1.

The physical average in the right-hand side of Eq. (14) can also be matched by the double statistical average

$$\overline{\cos^2\bigl(\omega_0 t + \phi(t)\bigr)} = \Bigl\langle \mathrm{E}\Bigl[\cos^2\bigl(\omega_0 t + \phi(t)\bigr)\Bigr]\Bigr\rangle . \tag{15}$$

Since the averages are linear operations, one has

$$\Bigl\langle \mathrm{E}\Bigl[\cos^2\bigl(\omega_0 t + \phi(t)\bigr)\Bigr]\Bigr\rangle = \frac{1}{2} + \frac{1}{2}\Bigl\langle \mathrm{E}\Bigl[\cos\bigl(2\omega_0 t + 2\phi(t)\bigr)\Bigr]\Bigr\rangle . \tag{16}$$

From the identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, one also has

$$\Bigl\langle \mathrm{E}\Bigl[\cos\bigl(2\omega_0 t + 2\phi(t)\bigr)\Bigr]\Bigr\rangle = \Bigl\langle \mathrm{E}\Bigl[\cos(2\omega_0 t)\cos\bigl(2\phi(t)\bigr)\Bigr]\Bigr\rangle - \Bigl\langle \mathrm{E}\Bigl[\sin(2\omega_0 t)\sin\bigl(2\phi(t)\bigr)\Bigr]\Bigr\rangle . \tag{17}$$

For the statistical average according to the density $p_\phi(\cdot)$, one has $\mathrm{E}[\cos(2\omega_0 t)\cos(2\phi(t))] = \cos(2\omega_0 t)\,\mathrm{E}[\cos(2\phi(t))]$, with $\mathrm{E}[\cos(2\phi(t))]$ now a deterministic constant independent of $t$, so that $\langle\cos(2\omega_0 t)\,\mathrm{E}[\cos(2\phi(t))]\rangle = \langle\cos(2\omega_0 t)\rangle\,\mathrm{E}[\cos(2\phi(t))]$. Also $\cos(2\omega_0 t)$ is a deterministic factor whose average over $t$ in the sense of the temporal Poisson process reduces to a uniform time average of the cosine function, which vanishes. One therefore has $\langle\cos(2\omega_0 t)\rangle = 0$, leading to $\langle\mathrm{E}[\cos(2\omega_0 t)\cos(2\phi(t))]\rangle = 0$. And in the same way in Eq. (17) one also has $\langle\mathrm{E}[\sin(2\omega_0 t)\sin(2\phi(t))]\rangle = 0$. The right-hand side of Eq. (17) is then zero, and therefore the right-hand side of Eqs. (15) and (16) is $1/2$, and this without any restrictive condition on the density $p_\phi(\cdot)$ or the correlation duration $\tau_c$.

One arrives at the same result by reasoning on the way is performed, from a very long realization of the random signal, a physical average like $\overline{\cos(2\omega_0 t)\cos(2\phi(t))}$. For all the values of $t$ congruent modulo $T_0$ to a same point $t_{\rm red}$ of the interval $[0, T_0[$, the term $\cos(2\omega_0 t)$ keeps the same value $\cos(2\omega_0 t_{\rm red})$, while the associated term $\cos(2\phi(t))$ visits all the values accessible to it with the statistical weight $p_\phi(\phi)$. The accumulation of all these products gives $\cos(2\omega_0 t_{\rm red})\,{\rm E}[\cos(2\phi)]$. The factor ${\rm E}[\cos(2\phi)]$ independent of $t_{\rm red}$ makes that the integral in $t_{\rm red} \in [0, T_0[$ reduces to the integral of a cosine which is identically zero. By this process, the physical average of the left-hand side of Eq. (15), after its linearization, is found to be 1/2.

It then follows that for the partially coherent wave, the intensity $I_0$ defined by Eq. (14) is $I_0 = A^2/2$. This is a familiar expression for the intensity, which is here connected to an explicit mathematical formulation of the partially coherent wave with arbitrary phase density $p_\phi(\cdot)$. Especially, the present derivation shows that the intensity of the wave is unaffected by the probability density $p_\phi(\cdot)$ of the random phase. This will no longer be the case for the coherence of the wave, as measured in an interference phenomenon, as we shall see next.

### 3.2. *Interference phenomenon*

With the partially coherent wave of Eq. (9), we return to the interference experiment of Section 2. The detector now receives the superposition

$$u_{\rm tot}(t) = u(t) + u(t+T) = A\cos\big(\omega_0 t + \phi(t)\big) + A\cos\big(\omega_0 t + \omega_0 T + \phi(t+T)\big)\,. \tag{18}$$

The intensity $I$ associated to the wave $u_{\rm tot}(t)$ of Eq. (18) and that the detector measures is

$$I = \lim_{T_{\rm int}\to\infty} \frac{1}{T_{\rm int}} \int_0^{T_{\rm int}} u_{\rm tot}^2(t)dt = \overline{u_{\rm tot}^2(t)}\,, \tag{19}$$

or

$$I = \overline{u^2(t)} + \overline{u^2(t+T)} + 2\,\overline{u(t)u(t+T)}\,. \tag{20}$$

By the same reasoning that terminates Section 3.1 after Eq. (13), one finds $\overline{u^2(t)} = \overline{u^2(t+T)} = I_0 = A^2/2$. The behavior in the interference of the intensity $I$ of Eq. (20) is then dependent on the autocorrelation function of the field

$$\overline{u(t)u(t+T)} = A^2 \times C(T)\,, \tag{21}$$

with the interference factor

$$C(T) = \overline{\cos\big(\omega_0 t + \phi(t)\big)\cos\big(\omega_0 t + \omega_0 T + \phi(t+T)\big)}\,. \tag{22}$$

giving the intensity

$$I = 2I_0[1 + 2C(T)]\,. \tag{23}$$

The physical average in the right-hand side of Eq. (22) can also be matched by the double statistical average

$$C(T) = \Big\langle {\rm E}\Big[\cos\big(\omega_0 t + \phi(t)\big)\cos\big(\omega_0 t + \omega_0 T + \phi(t+T)\big)\Big]\Big\rangle\,, \tag{24}$$

for which again two possible configurations have to be considered:

(i) When $\phi(t) = \phi(t+T) = \varphi_1$, with $\varphi_1$ a random number distributed according to the density $p_\phi(\cdot)$, then Eq. (24) gives

$$C(T) = \Big\langle \mathrm{E}\Big[\cos(\omega_0 t + \varphi_1)\cos(\omega_0 t + \omega_0 T + \varphi_1)\Big]\Big\rangle \tag{25}$$

$$= \frac{1}{2}\Big\langle \mathrm{E}\Big[\cos(\omega_0 T)\Big]\Big\rangle + \frac{1}{2}\Big\langle \mathrm{E}\Big[\cos(2\omega_0 t + \omega_0 T + 2\varphi_1)\Big]\Big\rangle . \tag{26}$$

With the same considerations that have been used to evaluate Eq. (17), one can conclude for Eq. (26) that $\langle \mathrm{E}[\cos(2\omega_0 t + \omega_0 T + 2\varphi_1)]\rangle = 0$ and $\langle \mathrm{E}[\cos(\omega_0 T)]\rangle = \cos(\omega_0 T)$. Therefore, in this configuration (i) one finds

$$C(T) = \frac{1}{2}\cos(\omega_0 T) . \tag{27}$$

(ii) When $\phi(t) = \varphi_1$ and $\phi(t+T) = \varphi_2$, with $\varphi_1$ and $\varphi_2$ two independent random numbers distributed according to the density $p_\phi(\cdot)$, then Eq. (24) gives

$$C(T) = \Big\langle \mathrm{E}\Big[\cos(\omega_0 t + \varphi_1)\cos(\omega_0 t + \omega_0 T + \varphi_2)\Big]\Big\rangle . \tag{28}$$

For the product of the two cosines in Eq. (28), one can use

$$\cos(\omega_0 t + \varphi_1) = \cos(\omega_0 t)\cos(\varphi_1) - \sin(\omega_0 t)\sin(\varphi_1) , \tag{29}$$

and

$$\cos(\omega_0 t + \omega_0 T + \varphi_2) = \cos(\omega_0 t + \omega_0 T)\cos(\varphi_2) - \sin(\omega_0 t + \omega_0 T)\sin(\varphi_2) . \tag{30}$$

And through Eqs. (29)–(30), the product of the two cosines in Eq. (28) develops into four product terms. One of wich is

$$C_1 = \Big\langle \mathrm{E}\Big[\cos(\omega_0 t)\cos(\varphi_1)\cos(\omega_0 t + \omega_0 T)\cos(\varphi_2)\Big]\Big\rangle , \tag{31}$$

which, via the two separate averages, gives

$$C_1 = \Big\langle \cos(\omega_0 t)\cos(\omega_0 t + \omega_0 T)\Big\rangle \mathrm{E}^2[\cos(\phi)] . \tag{32}$$

By the identity $2\cos(a)\cos(b) = \cos(a-b) + \cos(a+b)$, one gets for the temporal average in Eq. (32),

$$\Big\langle \cos(\omega_0 t)\cos(\omega_0 t + \omega_0 T)\Big\rangle = \frac{1}{2}\Big\langle \cos(\omega_0 T)\Big\rangle + \frac{1}{2}\Big\langle \cos(2\omega_0 t + \omega_0 T)\Big\rangle . \tag{33}$$

Since $\langle \cos(2\omega_0 t + \omega_0 T)\rangle = 0$, one finally obtains

$$C_1 = \frac{1}{2}\cos(\omega_0 T)\,\mathrm{E}^2[\cos(\phi)] . \tag{34}$$

Next, the second product term from Eq. (28) and accompanying $C_1$ of Eq. (31) is

$$C_2 = \Big\langle \mathrm{E}\Big[\sin(\omega_0 t)\sin(\varphi_1)\sin(\omega_0 t + \omega_0 T)\sin(\varphi_2)\Big]\Big\rangle \,. \tag{35}$$

Repeating for $C_2$ similar arguments as for $C_1$, one finally obtains

$$C_2 = \frac{1}{2}\cos(\omega_0 T)\,\mathrm{E}^2[\sin(\phi)]\,. \tag{36}$$

Again with similar arguments, the two last product terms accompanying $C_1$ and $C_2$ are

$$\begin{aligned} C_3 &= -\Big\langle \mathrm{E}\Big[\cos(\omega_0 t)\cos(\varphi_1)\sin(\omega_0 t + \omega_0 T)\sin(\varphi_2)\Big]\Big\rangle \\ &= -\frac{1}{2}\sin(\omega_0 T)\,\mathrm{E}[\cos(\phi)]\,\mathrm{E}[\sin(\phi)]\,, \end{aligned} \tag{37}$$

and

$$\begin{aligned} C_4 &= -\Big\langle \mathrm{E}\Big[\sin(\omega_0 t)\sin(\varphi_1)\cos(\omega_0 t + \omega_0 T)\cos(\varphi_2)\Big]\Big\rangle \\ &= \frac{1}{2}\sin(\omega_0 T)\,\mathrm{E}[\cos(\phi)]\,\mathrm{E}[\sin(\phi)]\,. \end{aligned} \tag{38}$$

By collecting the four product terms $C_1$ to $C_4$, Eq. (28) for configuration (ii) leads to

$$C(T) = \frac{1}{2}\cos(\omega_0 T)\Big(\mathrm{E}^2[\cos(\phi)] + \mathrm{E}^2[\sin(\phi)]\Big)\,. \tag{39}$$

Since over time, according to the temporal Poisson process, configuration (i) occurs with probability $\exp(-T/\tau_c)$ and configuration (ii) with probability $1 - \exp(-T/\tau_c)$, a weighted average of Eqs. (27) and (39) leads for Eqs. (22) and (24) to

$$C(T) = \frac{1}{2}\cos(\omega_0 T)\left[\exp\Big(-\frac{T}{\tau_c}\Big) + \Big(\mathrm{E}^2[\cos(\phi)] + \mathrm{E}^2[\sin(\phi)]\Big)\Big(1 - \exp\Big(-\frac{T}{\tau_c}\Big)\Big)\right]. \tag{40}$$

This determines the intensity $I$ of Eq. (23) at pathlength difference $T$ in the interference process. For this intensity pattern, the visibility at pathlength difference $T$, as defined by Eq. (6), now comes out as

$$\mathcal{V} = \exp\Big(-\frac{T}{\tau_c}\Big) + \Big(\mathrm{E}^2[\cos(\phi)] + \mathrm{E}^2[\sin(\phi)]\Big)\Big(1 - \exp\Big(-\frac{T}{\tau_c}\Big)\Big)\,. \tag{41}$$

We have with Eq. (41) a measure of the coherence of the wave exhibiting its dependence with the second-order statistics of the phase through the correlation duration $\tau_c$, but also — what is new here — with its first-order statistics through $\mathrm{E}^2[\cos(\phi)] + \mathrm{E}^2[\sin(\phi)]$ controlled by the probability density $p_\phi(\cdot)$.

### 3.3. *The classic case of a uniform phase*

As indicated in Section 3.1, the model of partially coherent wave is based on an empirical phenomenological description for the phase $\phi(t)$ at the microscopic level of elementary radiators. We now want to get some appreciation of the ability of this model to produce verifiable predictions at the macroscopic level of observable light. The common situation for a partially coherent light is a phase uniform over $[-\pi, \pi]$. In this case of a density $p_\phi(\phi)$ uniform over $[-\pi, \pi]$, the model has the expectations $\mathrm{E}[\cos(\phi)] = \mathrm{E}[\sin(\phi)] = 0$ and Eq. (40) simplifies into

$$C(T) = \frac{1}{2}\cos(\omega_0 T)\exp\Big(-\frac{T}{\tau_c}\Big)\,. \tag{42}$$

From Eq. (42), and coming back to Eq. (21), the model provides an expression of the autocorrelation function of the field, for any time delay $\tau$, as

$$\overline{u(t)u(t+\tau)} = \frac{A^2}{2}\cos(\omega_0 \tau)\exp\Big(-\frac{|\tau|}{\tau_c}\Big)\,. \tag{43}$$

Fourier transforming the autocorrelation of Eq. (43) gives access to the power spectrum of the wave, which comes out as a spectral line centered at frequency $\omega_0$, with a Lorentzian shape of width measured by $1/\tau_c$. The Lorentzian spectral line, as predicted by the model, is a spectrum which is very commonly observed for partially coherent monochromatic light [8]. With a correlation duration $\tau_c$ of the phase which goes to infinity, the model describes a completely coherent wave, with a field autocorrelation $\overline{u(t)u(t+\tau)} = A^2\cos(\omega_0\tau)/2$ associated to a Dirac-delta spectrum. By reduction of the correlation duration $\tau_c$, the model describes a degradation of the coherence, with a spectrum which broadens into a Lorentzian spectral line with width $1/\tau_c$ increasing as the coherence is reduced. These properties of the model match experimental properties commonly observable on partially coherent light.

For $p_\phi(\phi)$ uniform over $[-\pi, \pi]$, Eq. (41) of the model leads to an interferogram visibility

$$\mathcal{V} = \exp\Big(-\frac{T}{\tau_c}\Big)\,. \tag{44}$$

From Eq. (44), at any $T \ll \tau_c$, the model predicts $\mathcal{V} \approx 1$, identifying a situation of high coherence; and at any $T \gg \tau_c$, the model predicts $\mathcal{V} \approx 0$, identifying a situation of low coherence. These predictions of the model again match the common experimental observation in an interference phenomenon: the coherence measured by $\mathcal{V}$ is high at pathlength difference $T$ much smaller than the correlation duration $\tau_c$ and low at $T$ much larger than $\tau_c$. Also, the model via Eq. (44), at fixed $\tau_c$ imposed by the light source, describes the way the interferogram fades away with decreasing visibility $\mathcal{V}$, as the distance to the central fringe increases. Alternatively, at a given point with fixed $T$, varying the correlation duration $\tau_c$ of the phase, allows one to obtain at this point, according to the value assigned to $\mathcal{V}$ by Eq. (44), an interference of highly coherent waves when $\tau_c \gg T$, or of quasi incoherent waves when $\tau_c \ll T$, or any partial degree of coherence for intermediate $\tau_c$. These properties of the model reproduce the common behavior observed in a interference experiment.

Therefore, in the classic case of a phase uniform over $[-\pi, \pi]$, the present model of a partially coherent wave is able to yield predictions in accordance with common observable properties of light. This ability of the model based on a simple empirical description of the phase at the microscopic level, may be due to the fact that the model is used to derive predictions for macroscopic quantities resulting from statistical and temporal averages. As it is often the case in statistical physics, macroscopic quantities emerging from averages are not critically influenced by the detailed assumptions at the microscopic level, providing they capture a few essential basic features. The verifications of this Section suggest that this may be the case for the present model. Based on this grounding, we will now apply this model to explore the nonclassic case of a nonuniform random phase.

### 3.4. *Influence of the statistics of the phase*

We now examine the evolution of the degree of coherence measured by the visibility $\mathcal{V}$ of Eq. (41), in the general case of an arbitrary probability density $p_\phi(\cdot)$ for the phase.

At fixed $\tau_c > 0$, if $p_\phi(\phi)$ tends to the Dirac delta function $\delta(\phi)$ (the values of $\phi$ concentrate more and more around zero), then $\mathrm{E}[\cos(\phi)] \to 1$ and $\mathrm{E}[\sin(\phi)] \to 0$. It follows that in Eq. (41) the visibility $\mathcal{V} \to 1$, and one is returned to the situation of complete coherence of Eq. (7) whatever $T$ and $\tau_c$.

If the density $p_\phi(\phi)$ is uniform over $\phi \in [-\varphi_M, \varphi_M] \subseteq [-\pi, \pi]$, then $\mathrm{E}[\cos(\phi)] = \sin(\varphi_M)/\varphi_M$ and $\mathrm{E}[\sin(\phi)] = 0$. The case $\varphi_M \to 0$ gives the preceding case of the Dirac delta, while the case $\varphi_M = \pi$ is the classic case discussed in Section 3.3. With an arbitrary $\varphi_M \in [0, \pi]$ then

$$\mathcal{V} = \exp\Big(-\frac{T}{\tau_c}\Big) + \mathrm{sinc}^2(\varphi_M)\Big(1 - \exp\Big(-\frac{T}{\tau_c}\Big)\Big) . \tag{45}$$

At fixed $\tau_c$ and $T$, when the phase dispersion $\varphi_M$ increases from 0 to $\pi$, it results that $\mathcal{V}$ of Eq. (45) monotonically decreases from $\mathcal{V} = 1$ (complete coherence) to $\mathcal{V} = \exp(-T/\tau_c)$ (a state of partial coherence). One thus registers a monotonic loss of coherence, at fixed $\tau_c$ and $T$, when the phase dispersion $\varphi_M$ increases from 0 to $\pi$.

Yet, this monotonic loss of coherence is not necessarily the rule, for another phase distribution $p_\phi(\cdot)$, and this is a specific finding of this paper. For a simple illustration, we consider the density $p_\phi(\phi) = [\delta(\phi + \sigma_\phi) + \delta(\phi - \sigma_\phi)]/2$ describing a phase $\phi(t)$ randomly flipping between two discrete values $\pm\sigma_\phi$. From Eq. (41) one then gets

$$\mathcal{V} = \exp\Big(-\frac{T}{\tau_c}\Big) + \cos^2(\sigma_\phi)\Big(1 - \exp\Big(-\frac{T}{\tau_c}\Big)\Big) . \tag{46}$$

At fixed $\tau_c$ and $T$, when the phase dispersion $\sigma_\phi$ increases from 0 to $\pi$, it results that $\mathcal{V}$ of Eq. (46) first decreases from $\mathcal{V} = 1$ at $\sigma_\phi = 0$ (complete coherence) down to the minimum $\mathcal{V} = \exp(-T/\tau_c)$ when $\sigma_\phi = \pi/2$ (a state of partial coherence), to rise again up to $\mathcal{V} = 1$ at $\sigma_\phi = \pi$, restoring complete coherence. For the coherence measured by the visibility $\mathcal{V}$ at a fixed point $T$ and fixed correlation duration $\tau_c$ at the source, one thus registers a nonmonotonic variation, with a possibility of increasing the coherence by means of an increase in the phase dispersion $\sigma_\phi$, over certain ranges. This is a constructive effect (increased coherence) obtained through an enhancement of a noise component (increased phase dispersion). This type of behavior is reminiscent of stochastic resonance or useful-noise effects, by which an increase in the noise level improves the performance, as reported for instance in [12–16, 18, 19, 21–23]. Especially, [21–23] showed the possibility of stochastic resonance with phase noise on a periodic wave, although with no reference to optical coherence.

Based on the results of [10, 23], it can be expected that other distributions $p_\phi(\cdot)$ of the phase might lend themselves to a form of nonmonotonic improvement of the coherence. For a more elaborate distribution, we consider, in similarity with [10,23], the mixture of two Gaussians

$$p_\phi(\phi) = \frac{1}{\Delta\phi\, 2\sqrt{2\pi}} \exp\left(-\frac{(\phi+\phi_0)^2}{2\,\Delta\phi^2}\right) + \frac{1}{\Delta\phi\, 2\sqrt{2\pi}} \exp\left(-\frac{(\phi-\phi_0)^2}{2\,\Delta\phi^2}\right). \qquad (47)$$

The probability density of Eq. (47) is made of two Gaussian peaks of same width $\Delta\phi$, one centered at $-\phi_0$ and the other at $\phi_0$. This density of Eq. (47) has mean $m_\phi = 0$ and variance $\sigma_\phi^2 = \phi_0^2 + \Delta\phi^2$; and it leads to the averages $\mathrm{E}[\sin(\phi)] = 0$ and $\mathrm{E}[\cos(\phi)] = \cos(\phi_0)\exp(-\Delta\phi^2/2)$. The resulting interferogram visibility from Eq. (41) is then

$$\mathcal{V} = \exp\Big(-\frac{T}{\tau_c}\Big) + \cos^2(\phi_0)\exp(-\Delta\phi^2)\Big(1 - \exp\Big(-\frac{T}{\tau_c}\Big)\Big). \qquad (48)$$

At fixed width $\Delta\phi$, increasing $\phi_0$ increases the rms phase dispersion $\sigma_\phi = (\phi_0^2 + \Delta\phi^2)^{1/2}$. The resulting evolution of the visibility $\mathcal{V}$ of Eq. (48) is presented in Fig. 3. The evolution of Fig. 3 displays again a possibility of improving the coherence measured by $\mathcal{V}$ by means of an increase of the phase disorder measured by its rms dispersion $\sigma_\phi$.

Another nonmonotonic evolution of the visibility $\mathcal{V}$ can also be obtained with a density $p_\phi(\cdot)$ uniform over $[-\varphi_M, \varphi_M]$ for the phase, leading to $\mathcal{V}$ of Eq. (45). When the phase dispersion $\varphi_M$ increases above $\pi$, in Eq. (45) the factor $\mathrm{sinc}^2(\varphi_M)$ starts to rise again, leading to an increasing visibility $\mathcal{V}$ in this range of $\varphi_M$, as represented in Fig. 4.

In Fig. 4, when the dispersion $\varphi_M$ of the uniform random phase increases, a nonmonotonic evolution of the visibility $\mathcal{V}$ is observed for any value of $T/\tau_c$. When $\varphi_M$ increases above $\pi$ in Fig. 4, the resulting improvement of $\mathcal{V}$ is relatively small compared to the visibility of $\mathcal{V} = 1$ with no phase dispersion at $\varphi_M = 0$. However, common optical conditions will often impose a uniform random phase with dispersion $\varphi_M = \pi$. This value $\varphi_M = \pi$ is precisely the location of a minimum of the visibility $\mathcal{V}$ in Fig. 4, for any $T/\tau_c$. At $T/\tau_c \gg 1$, this minimum of the visibility is very close to zero, as seen for instance in Fig. 4 at $T/\tau_c = 10$ and $\varphi_M = \pi$. From this configuration, if the phase dispersion is increased up to $\varphi_M \approx 3\pi/2$, the relative increase of the visibility is then strong. In Fig. 4 at $T/\tau_c = 10$, we have $\mathcal{V}(\varphi_M = \pi) \approx 5 \times 10^{-5}$ and $\mathcal{V}(\varphi_M \approx 3\pi/2) \approx 5 \times 10^{-2}$, which corresponds to an amplification of three orders of magnitude of $\mathcal{V}$ when the phase dispersion is

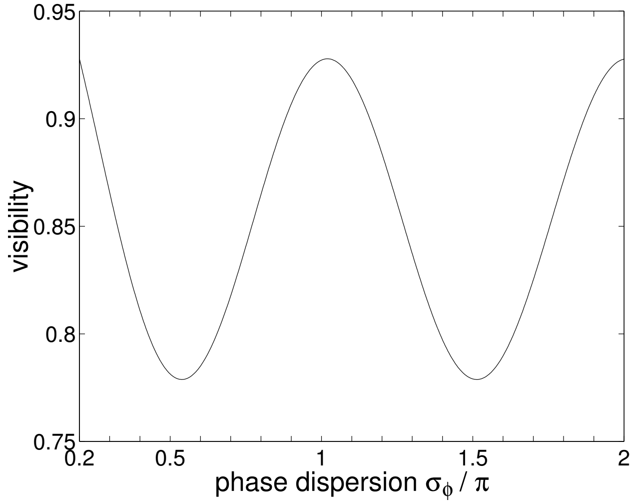


Fig. 3. Interferogram visibility $\mathcal{V}$ of Eq. (48), as a function of the phase dispersion $\sigma_\phi = (\phi_0^2 + \Delta\phi^2)^{1/2}$, by varying $\phi_0$ at $\Delta\phi = \pi/5$ and $T/\tau_c = 1/4$.

increased from $\varphi_M = \pi$ to $\varphi_M \approx 3\pi/2$. Knowing that common optical conditions are likely to impose $\varphi_M = \pi$ to begin with, if an action is practically implementable to further increase the phase dispersion up to $\varphi_M \approx 3\pi/2$, this could translate into strong enhancement of the coherence measured by $\mathcal{V}$. We note that it is in principle quite possible to have a random phase $\phi(t)$ varying over an interval $[-\varphi_M, \varphi_M]$ ranging beyond $[-\pi, \pi]$. This could be the case with mechanical vibrations producing pathlength fluctuations that would exceed a sufficient amount relative to the wavelength, as realized in [24] for instance. This is the same possibility which is present with the phase noise in [10, 23], and also all the techniques dealing with phase wrapping and unwrapping are in fact related to this same issue.

If we return to the general expression of the visibility $\mathcal{V}$ in Eq. (41), one always has $0 \leq \mathrm{E}^2[\cos(\phi)] + \mathrm{E}^2[\sin(\phi)] \leq 1$, for any density $p_\phi(\cdot)$. It results that, at given density $p_\phi(\cdot)$ and fixed pathlength difference $T$, the visibility $\mathcal{V}$ of Eq. (41) always decreases when the correlation duration $\tau_c$ decreases. Decreasing the correlation duration $\tau_c$ of the phase $\phi(t)$ is one way of increasing the phase disorder, but this way of increasing the phase disorder can never improve the coherence, in the wave model of Eq. (9). On the contrary, as demonstrated above, increasing the phase disorder by increasing the phase dispersion is capable of improving the coherence.

## 4. Discussion

We have analyzed a simple model of a scalar optical wave with partial coherence. The model is specially devised to describe the influence on the coherence, of both first-order and second-order statistical properties of the random phase. Classic models of partially coherent light [8] do not allow direct control on the probability density of the phase as a free parameter. The model of Section 3.1 was implemented

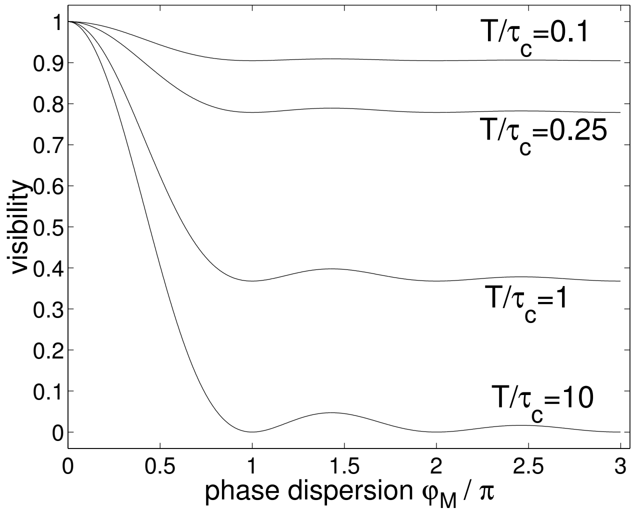


Fig. 4. Interferogram visibility $\mathcal{V}$ of Eq. (45), as a function of the phase dispersion $\varphi_M$, for different values of $T/\tau_c$.

in order to obtain such a control, with the aim of theoretically investigating the impact of the probability density of the phase on the coherence. For the classic case of a random phase uniform over $[-\pi, \pi]$, the model predicts common behaviors in accordance with observable properties of light. Based on this grounding, we have applied the model to theoretically explore the impact of nonclassic statistics of the phase on the coherence. In this way, the model shows that increasing the disorder of the wave by decreasing the correlation duration of the random phase always degrades the coherence of the wave; this is a known and expected behavior of the optical coherence, captured by the model. Furthermore, the model reveals that increasing the disorder of the wave by increasing its phase dispersion can sometimes lead to an improvement of the coherence of the wave. This behavior is reminiscent of stochastic resonance or similar useful-noise effects, where an increase in the amount of noise or disorder can improve some measure of order or performance or efficacy of processing. Usually, stochastic resonance studies assume additive noise; a few studies have exhibited stochastic resonance with phase noise [21–23] although with no reference to optical coherence. The presence of phase noise is natural to optical waves, and a standard measure of order in this context, the optical coherence, is shown here to lend itself to a form of stochastic resonance. The present results demonstrating a theoretical possibility of noise-improved optical coherence, open a new direction for stochastic resonance investigations. For instance, the present model of partially coherent wave could serve as a basis for theoretical analysis of more complex optical experiments, for example frequency doubling or other nonlinear effects, in order to examine whether it is possible to register phenomena where a measure of efficacy (other than the visibility $\mathcal{V}$) could be maximized in the presence of a partial coherence, away from both complete coherence and complete incoherence.

It is interesting to note that a different but related theoretical exploration of the impact of the probability density of the phase, can be found in the recent publication [25]. Reference [25] studies sums of random phasors to model optical speckle, and it observes that the contrast of a partially developed speckle can be changed, and specially increased, by varying the probability density of the phase of random phasors. In similarity with our present study, [25] gives another theoretical study manifesting some possibly beneficial influence of varying an optical phase away from uniformity over $[-\pi, \pi]$.

If we follow the present theoretical analysis, it suggests that there might be some benefit in trying to vary, away from uniformity over $[-\pi, \pi]$, the distribution of the phase of a partially coherent wave. A subsequent issue will be to consider the practical possibilities for varying the phase. The useful approach though, would not be to try to experimentally reproduce the precise conditions of our empirical model of Section 3.1, but rather to consider that our model suggests, through the consideration of simple theoretical conditions, some possible benefit of varying the distribution of the phase; from here then, any practical means of varying the phase could be interesting to examine in this respect. Some proposals to be examined could be as follows. Some control on the phase distribution could be obtained by imposing, with electroacoustic devices, random vibrations interacting with the wave, or pathlength fluctuations (for instance with piezoelectric actuators) for one or both of the interfering waves relative to the other. Magneto-optical or electro-optical phase-modulating devices could also be tested. Concentration of a noisy wavefront obtained from speckle noise could also be examined. All these, or other, proposals to gain some control on the phase distribution of a partially coherent wave, remain to be practically investigated.

The results reported here concerning the behavior of wave coherence, might also be relevant outside optics, with radiowaves or acoustics, for coherent or interferometric imaging applications for instance. In another direction, connection with the coherence of the quantum wave function could also be envisaged [26, 27], and the impact of noise on the decoherence process in quantum information processing.

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