Generation of signals with long-range correlations

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The authors propose a model in the form of a first-order recurrence which is capable of generating, over potentially unlimited ranges, long-range correlations of power-law decay with a controllable exponent.

Long-range correlations in random signals are identified by a slow (slower than exponential) decay. Typically, these correlations decay according to a power law, conferring statistical self-similarity and a fractal character to the signals [1]. Such random signals are experimentally observed in a wide variety of physical processes, including telecommunication or motorway traffic or noise in semiconductors [1]. The theoretical modelling and practical synthesis of such signals with long-range correlations remain important issues, which have not been fully resolved. The few models available, among which are fractional Brownian motions [1], are in principle of infinite order. For practical synthesis, they have to be truncated, which restricts the long-range correlations to limited ranges. Other synthesis methods, such as Cholesky decomposition or wavelet expansions [1], perform block synthesis instead of recurrent synthesis. When a realisation of N points is synthesised, the subsequent addition of one more point with long-range correlations usually requires a new synthesis of a complete block of $N + 1$ points from scratch. In this Letter, we present a model, in the form of a first-order recurrence, which is capable of performing on-line synthesis of long-range correlations over potentially unlimited ranges. A simpler earlier version, with no control of the exponent of the power-law correlations, was numerically investigated in [2]; we extend the model here to provide this control and also add theoretical elements.

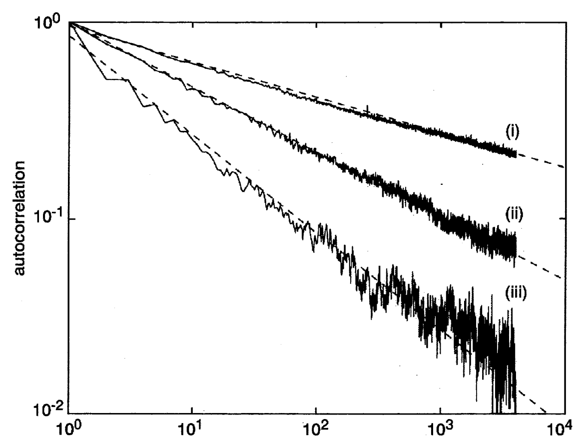

Fig. 1 *Normalised autocorrelation function of y(k) numerically evaluated (solid lines) with regression lines (dashed)*

(i) $b = 0.1$, $-\beta = -0.18$
(ii) $b = 0.2$, $-\beta = -0.32$
(iii) $b = 0$, $-\beta = -0.50$

The algorithm is shown below. $x(k)$ is an input sequence of independent and identically distributed random variables with zero mean. $y(k)$ is the output signal exhibiting power-law correlations with exponent β. The parameter b of the nonstationary gain $g(k)$ of eqn. 1 will provide control over β.

Algorithm generating $y(k)$ with long-range correlations:

$X(0) = Y(0) = 0$
$k = 1$; $k_0 = 0$
Repeat

$$g(k) = (k - k_0)^b \qquad (1)$$

$$X(k) = X(k-1) + g(k)x(k) \qquad (2)$$

$$Y(k) = \max[Y(k-1), X(k)] \qquad (3)$$

$$y(k) = Y(k) - Y(k-1) \qquad (4)$$

If $y(k) > 0$ then $k_0 \leftarrow k$ End If
$k \leftarrow k + 1$
Until stopping is requested

The above algorithm lends itself to a direct numerical implementation. It makes possible a numerical evaluation of the autocorrelation function $E[y(k)y(k + \tau)]$, performed here by the empirical average $N^{-1}\Sigma_{k=1}^{N} y(k)y(k + \tau)$ over one realisation and with $N = 10^7$. The results of Fig. 1 reveal the power-law evolution $\sim\tau^{-\beta}$ with the lag τ.

An empirical law for controlling β from b is deduced in Fig. 2 from numerical evaluations of the autocorrelation function as in Fig. 1.

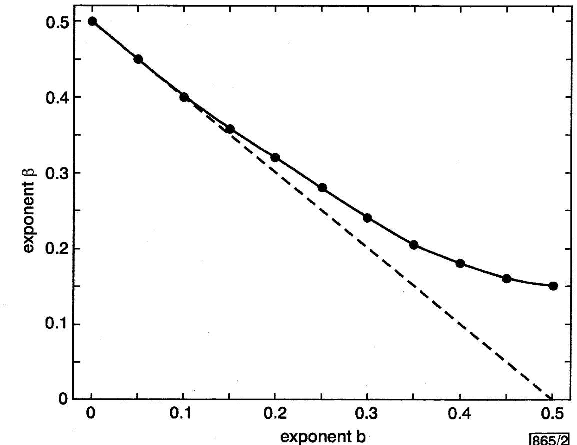

Fig. 2 *Exponent β of power-law correlations against b of gain g(k)*

—●— numerical evaluations
– – – – theory valid at low b

Theoretical arguments can also be given for establishing the power-law form of the correlations in $y(k)$. The random signal $y(k)$ represents the successive increments of the running maximum $Y(k)$ of the nonhomogeneous random walk $X(k)$ having increments $g(k)x(k)$. $y(k)$ is formed by intervals where $y(k) = 0$ interleaved with intervals where $y(k) > 0$, these intervals occurring over all time scales. According to eqn. 3 of the above algorithm, $y(k) > 0$ at each time k where the walk $X(k)$ realises a first passage. $y(k)$ also incorporates a renewal property, since according to eqn. 3 of the proposed algorithm, at each time k where $y(k) > 0$, one has $Y(k) = X(k)$, and for the subsequent evolution of the increment y it is just as if the process had been reset to its initial condition $Y = X = 0$. The dependence in the lag τ of $E[y(k)y(k + \tau)]$ is conveyed by the probability $\Pr\{y(k + \tau) > 0 | y(k) > 0\} = U(\tau)$, a function of τ only, due to the renewal property of $y(k)$. $U(\tau)$ can be expressed as

$$U(\tau) = \int_0^{+\infty} u(h, \tau) dh \qquad (5)$$

where $u(h, \tau)$ is the probability density for a first passage in h at time τ of the walk X started at $X = 0$ at time 0.

Based on the asymptotic properties of nonhomogeneous random walks [3], appropriate for the long-range behaviour in τ, we obtain

$$u(h, \tau) = A\tau^{2b} \frac{2h}{\sqrt{2\pi}\sigma_\tau^3} \exp\left(-\frac{h^2}{2\sigma_\tau^2}\right) \qquad (6)$$

where A is a constant proportional to the variance of the input $x(k)$, and $\sigma_\tau = \sqrt{[2A/(b + 1)]}\tau^{b+1/2}$. Integration of eqn. 6 according to eqn. 5 yields

$$U(\tau) = (1 + b)^{1/2} \sqrt{\frac{A}{\pi}} \tau^{b-1/2} \qquad (7)$$

This result thus predicts a power-law behaviour $\tau^{b-1/2}$ of $U(t) = \Pr\{y(k + \tau) > 0 | y(k) > 0\}$. This will translate into a power-law behaviour $\tau^{-\beta}$ of the autocorrelation $E[y(k)y(k + \tau)]$ with the same exponent $\beta = -b + 1/2$ only when the amplitude of $y(k + \tau)$ is not dependent on τ. This will occur strictly when $b = 0$ in the nonstationary gain, yielding $\beta = 1/2$ as verified numerically by Fig. 2. At $b > 0$, the increment $y(k + \tau)$ of the walk X between two successive first passages of X separated by a time τ, will grow as τ^b. This still preserves the power-law form of the correlation as visible in Fig. 1

at $b > 0$, but with an exponent β which gradually departs from $-b + 1/2$ as shown in Fig. 2 as b increases. Further, when b grows above 1/2 we have observed numerically that the long-range correlations are still preserved in $y(k)$ but with a decay in τ which is even slower than a power law.

The process $y(k)$ provides long-range correlations over potentially unlimited ranges by means of a simple first-order recurrent algorithm. Further, $y(k)$ can be used to trigger or modulate auxiliary random processes in many different ways, so as to add variability to the generated signals while preserving long-range correlations. Interesting applications can be found for simulation and performance evaluation in telecommunication networks and other fields.

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Stochastic rate and temporal codes with asymmetric bit errors

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The relative precision in estimating average pulse rates using stochastic temporal and rate codes is compared. In most situations, temporal codes exhibit less uncertainty than rate codes at low Bernoulli probabilities and *vice versa*. The balance is also affected by asymmetric $0 \rightarrow 1$ and $1 \rightarrow 0$ bit error probabilities.

Introduction: Stochastic signal processing [1] is one method of reducing both the power dissipation and the silicon area in circuit implementations of digital signal processors and artificial neural networks, while improving the fault tolerance and enabling variable-precision computations in fixed hardware [2 – 4]. Stochastic arithmetic and nonlinear operations employ simple logic gates and finite state machines [3, 4]. Hardware-efficient parallel pseudorandom number generators required for these implementations exploit cellular automata [5]. Limitations arise however from the statistical uncertainties in determining average pulse rates which represent the information [6]. In this Letter, we compare two approaches to stochastic signal processing.

Rate codes based on Bernoulli processes: When stochastic rate codes are adopted to represent information in a digital signal processing system, the pulse rate or signal value is deterniined by the number of counts or pulses k in a count interval of n clock cycles. In each clock cycle, the signals are governed by the Bernoulli probability p_0. This is a memoryless process. The probabilities of receiving a 1 or a 0 in any given clock cycle are

$$p(1) = p_0 \qquad p(0) = 1 - p_0 \tag{1}$$

The probability of receiving k pulses in a count interval of n clock cycles is given by the binomial distribution [7]

$$p(k) = C(n, k) p_0^k (1 - p_0)^{n-k} \tag{2}$$

for $0 \leq k \leq n$, and zero otherwise, where the binomial coefficients $C(n, k) = n!/(k! \cdot (n - k)!)$, which applies for an arbitrary value of Bernoulli probability $0 \leq p_0 \leq 1$. The mean and variance of this distribution are np_0 and $np_0(1 - p_0)$, respectively. To employ these codes, one must count pulses over an extended interval of time n. The coefficient of variation in the estimate of the mean rate decreases as $n^{-1/2}$.

In the presence of bit errors due to noise and other mechanisms, the Bernoulli probability p_0 is modified. Let us assume that the probability of recording a 0 when a pulse is actually present is given by η and the probability of receiving a 1 when no pulse is present is given by γ. Then the modified Bernoulli probability is

$$p_E = p_0(1 - \eta) + (1 - p_0)\gamma \tag{3}$$

Fig. 1 shows an example of the probability distributions $p(k)$ for rate codes, for various error conditions, when the error-free Bernoulli probability is $p_0 = 0.4$. Note that for case (i) in this Figure $p_E = p_0$. The abscissa corresponds to k counts in interval of $n = 16$ clock cycles. When $\eta > \gamma$ one experiences a reduced expected count, whereas for $\gamma > \eta$ the expected count increases. Note that the case of symmetrical error probabilities $\eta = \gamma = 0.2$ also results in a greater expected count than the error-free case. This is because $p_0 < 0.5$ so $0 \rightarrow 1$ errors slightly dominate over $1 \rightarrow 0$ errors. For $p_0 > 0.5$ there are on average more 1s than 0s and the reverse is true. We thus find that the nature of the error mechanisms can have a significant effect on the estimates of the true pulse rates.

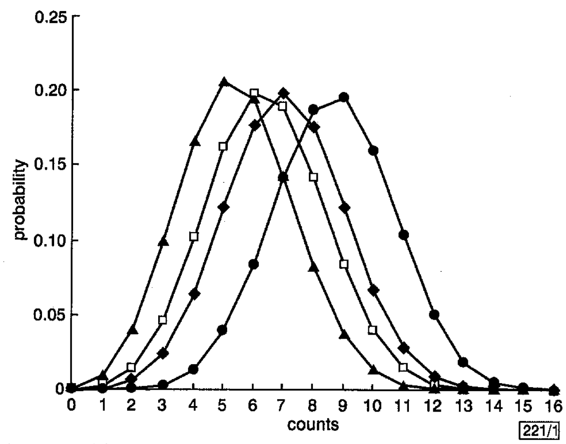

Fig. 1 *Modified binomial distributions of pulse counts for rate codes with asymmetrical bit errors*

$n = 16$, $p_0 = 0.4$
—□— $\eta = 0, \gamma = 0, p_E = p_0$
—◆— $\eta = 0.2, \gamma = 0.2$
—▲— $\eta = 0.3, \gamma = 0.1$
—●— $\eta = 0.1, \gamma = 0.3, p_E = p_0$

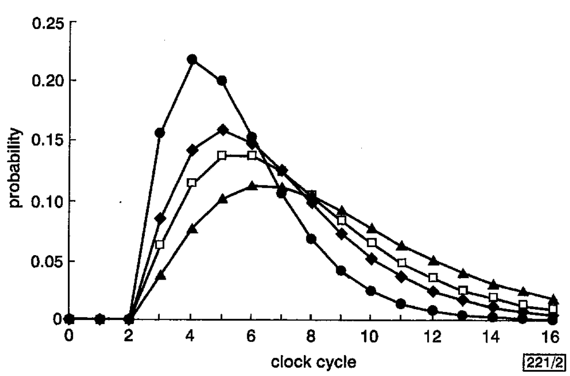

Fig. 2 *Modified negative binomial distributions of pulse counts for temporal codes (r = 3) with asymmetrical bit errors*

—□— $\eta = 0, \gamma = 0, p_E = p_0$
—◆— $\eta = 0.2, \gamma = 0.2$
—▲— $\eta = 0.3, \gamma = 0.1$
—●— $\eta = 0.1, \gamma = 0.3, p_E = p_0$

Temporal codes: Temporal codes have often been suggested as a means of expediting the determination of the pulse rate. In this approach, one determines the time to the first (in general the rth) pulse. This allows for a more rapid estimate of the mean rate than with the rate codes above, at the expense of increased variance in the estimate. The probability of the rth pulse arriving on the mth clock cycle is given by the negative binomial distribution [7]

$$p(m) = C(m - 1, r - 1) p_0^r (1 - p_0)^{m-r} \tag{4}$$