Digital Signal Processing 23 (2013) 1585–1591

Contents lists available at SciVerse ScienceDirect

Digital Signal Processing

www.elsevier.com/locate/dsp

Weak signal detection: Condition for noise induced enhancement

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ARTICLE INFO

Article history:
Available online 23 May 2013

Keywords:
Noise induced enhancement
Noise-enhanced detection
Stochastic resonance
Signal detection

ABSTRACT

For the detection of a weak known signal in additive white noise, a generalized correlation detector is considered. In the case of a large number of measurements, an asymptotic efficacy is analytically computed as a general measure of detection performance. The derivative of the efficacy with respect to the noise level is also analytically computed. Positivity of this derivative is the condition for enhancement of the detection performance by increasing the level of noise. The behavior of this derivative is analyzed in various important situations, especially showing when noise-enhanced detection is feasible and when it is not.

1. Introduction

Recently, the employment of noise in enhancing the performance of signal processors has emerged as a topic of significant interest [1–11]. This notion is rooted in the concept of stochastic resonance (SR) that was first elucidated in the area of climate dynamics [12]. The attraction of SR is that an appropriate non-zero noise level can improve, rather than degrade, the performance of nonlinear systems [13–19]. So far, several static nonlinearities arising in various signal processing problems were shown to exhibit a noise-enhanced effect, such as quantizers [7–11] and nonlinear detectors [1–4,20–30]. Now, this method of enhancement via noise is still under investigation as a technique with useful potential for nonlinear signal processing.

In this letter, we focus on the detection enhancement of a weak signal in additive white noise by a generalized correlation detector. With a sufficiently large observation size, the detection performance of the detector is determined by the normalized asymptotic efficacy ξ_{GC} [31]. We show that both the efficacy ξ_{GC} and its derivative with respect to the noise level can be analytically computed. This derivative and its condition of positivity are analyzed in various important situations, allowing us to conclude when increasing the level of noise can improve the detection performance, and when it cannot. The result provides not only an easily implemented criterion for exploring the role of noise in detectors, but also the operational levels of noise that we can employ.

2. Noise enhancement of weak signal detection

2.1. Model

Consider the observation vector $X = (X_1, X_2, \ldots, X_N)$ of real-valued components X_n defined by

$$X_n = \theta s_n + Z_n, \quad n = 1, 2, \ldots, N, \tag{1}$$

where the components Z_n form a sequence of independent and identically distributed (i.i.d.) random variables with probability density function (PDF) f_z and variance σ_z^2, and the known signal components s_n have signal strength θ [31]. The average signal power satisfies $0 < P_s = \sum_{n=1}^{N} s_n^2/N < \infty$ [31]. The detection problem can be formulated as a hypothesis-testing problem for deciding a null hypothesis H_0 ($\theta = 0$) and an alternative hypothesis H_1 ($\theta > 0$) associated with the joint probability densities

$$H_0\colon \quad f_X(X) = \prod_{n=1}^{N} f_z(X_n) \quad \text{for } \theta = 0,$$

$$H_1\colon \quad f_X(X) = \prod_{n=1}^{N} f_z(X_n - \theta s_n) \quad \text{for } \theta > 0. \tag{2}$$

In order to decide H_0 or H_1 on the basis of X, consider a generalized correlation detector

$$T_{\mathrm{GC}}(X) = \sum_{n=1}^{N} g(X_n) s_n \underset{H_0}{\overset{H_1}{\gtrless}} \gamma, \tag{3}$$

where the memoryless nonlinearity g has zero mean under f_z, i.e. $\mathrm{E}_z[g(x)] = \int_{-\infty}^{\infty} g(x) f_z(x)\,dx = 0$ and the test threshold is γ [31]. In the asymptotic case of $\theta \to 0$ and for a sufficiently large

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1051-2004/$ – see front matter
http://dx.doi.org/10.1016/j.dsp.2013.05.009

observation size N, the test statistic $T_{\rm GC}$, according to the central limit theorem, converges to a Gaussian distribution with mean $\mathrm{E}_z[T_{\rm GC}|H_0] = 0$ and variance $\mathrm{var}[T_{\rm GC}|H_0] = \mathrm{E}_z[T_{\rm GC}^2|H_0] = NP_s\mathrm{E}_z[g^2(x)]$ under the null hypothesis H_0 [31]. Similarly, under the hypothesis H_1, $T_{\rm GC}$ is also asymptotically Gaussian with mean $\mathrm{E}_z[T_{\rm GC}|H_1] \approx \theta NP_s\mathrm{E}_z[g'(x)]$ and variance $\mathrm{var}[T_{\rm GC}|H_1] = \mathrm{var}[T_{\rm GC}|H_0]$ [31]. Here, the derivatives $g'(x) = dg(x)/dx$ and $f'_z(x) = df_z(x)/dx$ exist for almost all x. Given a false alarm probability $P_{\rm FA}$, the detection probability $P_{\rm D}$ of the generalized correlation detector can be expressed as

$$P_{\rm D} = \mathrm{Q}\big[\mathrm{Q}^{-1}(P_{\rm FA}) - \theta\sqrt{NP_s}\sqrt{\xi_{\rm GC}}\,\big] = \mathrm{Q}\left[\mathrm{Q}^{-1}(P_{\rm FA}) - \theta\sqrt{\sum_{n=1}^{N} s_n^2}\sqrt{\xi_{\rm GC}}\right], \tag{4}$$

where $\mathrm{Q}(x) = \int_x^\infty \exp[-t^2/2]/\sqrt{2\pi}\,dt$ and its inverse function is Q^{-1} [31]. Thus, for fixed N and θP_s (since the signal is known), $P_{\rm D}$ is a monotonically increasing function of the normalized asymptotic efficacy $\xi_{\rm GC}$ given by [31]

$$\xi_{\rm GC} = \lim_{N\to\infty} \frac{\left\{\frac{d\mathrm{E}_z[T_{\rm GC}(X)]}{d\theta}\big|_{\theta=0}\right\}^2}{P_s N\,\mathrm{var}[T_{\rm GC}(X)]|_{\theta=0}} = \frac{\mathrm{E}_z^2[g'(x)]}{\mathrm{E}_z[g^2(x)]} \leqslant \mathrm{E}_z\left[\frac{f_z'^2(x)}{f_z^2(x)}\right] = I(f_z), \tag{5}$$

where the expectation $\mathrm{E}_z[f_z'^2(x)/f_z^2(x)]$ is the Fisher information $I(f_z)$ of f_z, and the equality occurs as

$$g(x) = C\frac{f_z'(x)}{f_z(x)} \triangleq g_{\rm LO}(x), \tag{6}$$

by the Cauchy–Schwarz inequality for a constant C. Here, $g_{\rm LO}(x)$ represents the locally optimal nonlinearity [31].

It is noted that $P_{\rm D}$ of Eq. (4) is a monotonically increasing function of $\xi_{\rm GC}$. Thus, as the noise level σ_z increases, the positive derivative

$$\frac{\partial\xi_{\rm GC}}{\partial\sigma_z} > 0 \tag{7}$$

indicates the occurrence of the noise-enhanced detection phenomenon. When the inequality of Eq. (7) holds for $0 < \sigma_z < \sigma_z^{\rm opt}$ and the equality

$$\frac{\partial\xi_{\rm GC}}{\partial\sigma_z}\bigg|_{\sigma_z=\sigma_z^{\rm opt}} = 0 \tag{8}$$

has only one solution $\sigma_z = \sigma_z^{\rm opt}$, then $\sigma_z^{\rm opt}$ is the optimal noise level that maximizes $\xi_{\rm GC}$. It is noted that the signal strength θ is small enough to allow us to use the first-order approximations leading to the detection probability of Eq. (4), and the noise-enhanced detection performance indicated by Eq. (7) is valid for arbitrary small signal level $\theta > 0$.

In the following, we assume that the scaled noise $Z(t) = \sigma_z Z_0(t)$ has PDF $f_z(z) = f_{z_0}(z/\sigma_z)/\sigma_z$ and the cumulative distribution function $F_z(x) = F_{z_0}(z/\sigma_z)$ [10,31]. Here, $Z_0(t)$ has a standardized PDF f_{z_0} with unity variance $\sigma_{z_0}^2 = 1$, the cumulative distribution function is $F_{z_0}(x) = \int_{-\infty}^x f_{z_0}(u)\,du$ and the Fisher information $I(f_{z_0}) = \mathrm{E}_{z_0}[f_{z_0}'^2(x)/f_{z_0}^2(x)]$. Then, the Fisher information $I(f_z) = I(f_{z_0})/\sigma_z^2$.

2.2. Noise enhancement by noise tuning

Corollary 1. *No noise-enhanced detection phenomenon will occur in the locally optimal detector*

$$T_{\rm LO}(X) = \sum_{n=1}^{N} g_{\rm LO}(X_n)s_n \underset{H_0}{\overset{H_1}{\gtrless}} \gamma, \tag{9}$$

with the nonlinearity $g_{\rm LO}$ defined in Eq. (6).

Proof. From Eqs. (5) and (6), the locally optimal detector in Eq. (9) has the normalized asymptotic efficacy $\xi_{\rm LO} = I(f_z) > 0$. Then, for $\sigma_z > 0$, we have

$$\frac{\partial\xi_{\rm LO}}{\partial\sigma_z} = \frac{\partial I(f_z)}{\partial\sigma_z} = -\frac{2I(f_{z_0})}{\sigma_z^3} < 0. \tag{10}$$

Thus, no noise-enhanced detection phenomenon will occur. □

Corollary 2. *The dead-zone limiter detector*

$$T_{\rm DZ}(X) = \sum_{n=1}^{N} g_{\rm DZ}(X_n)s_n \underset{H_0}{\overset{H_1}{\gtrless}} \gamma, \tag{11}$$

employs the characteristic

$$g_{\rm DZ}(x) = \begin{cases} -1 & \text{for } x < -\lambda, \\ 0 & \text{for } -\lambda \leqslant x \leqslant \lambda, \\ +1 & \text{for } x > \lambda, \end{cases} \tag{12}$$

with response threshold $\lambda > 0$. Given the threshold λ, the noise-enhanced detection effect will occur in the interval $\sigma_z \in (0, \sigma_z^{\rm opt})$, where the optimal noise level $\sigma_z^{\rm opt}$ is the non-zero solution of

$$\frac{\sigma_z}{\lambda} = g_{\rm LO}^{z_0}\left(\frac{\lambda}{\sigma_z}\right) - \frac{f_{z_0}\left(\frac{\lambda}{\sigma_z}\right)}{2\left[1 - F_{z_0}\left(\frac{\lambda}{\sigma_z}\right)\right]}, \tag{13}$$

with the nonlinearity

$$g_{\rm LO}^{z_0}(x) = -\frac{f_{z_0}'(x)}{f_{z_0}(x)}. \tag{14}$$

Proof. From Eq. (5), the normalized asymptotic efficacy $\xi_{\rm DZ}$ of the dead-zone limiter detector is [31,32]

$$\xi_{\rm DZ} = \frac{\mathrm{E}_z^2[g_{\rm DZ}'(x)]}{\mathrm{E}_z[g_{\rm DZ}^2(x)]} = \frac{2f_z^2(\lambda)}{1 - F_z(\lambda)}. \tag{15}$$

Since

$$\frac{\partial F_z(\lambda)}{\partial\sigma_z} = \frac{\partial F_{z_0}(\lambda/\sigma_z)}{\partial\sigma_z} = -\frac{\lambda f_{z_0}(\lambda/\sigma_z)}{\sigma_z^2} = -\frac{\lambda f_z(\lambda)}{\sigma_z}, \tag{16}$$

we obtain

$$\frac{\partial\xi_{\rm DZ}}{\partial\sigma_z} = \frac{4f_z(\lambda)\frac{\partial f_z(\lambda)}{\partial\sigma_z}[1 - F_z(\lambda)] - 2f_z^2(\lambda)f_z(\lambda)\frac{\lambda}{\sigma_z}}{[1 - F_z(\lambda)]^2} \geqslant 0, \tag{17}$$

$$\Rightarrow \quad \frac{\partial f_z(\lambda)}{\partial\sigma_z} - \frac{\lambda}{\sigma_z}\frac{f_z^2(\lambda)}{2[1 - F_z(\lambda)]} \geqslant 0, \tag{18}$$

$$\Rightarrow \quad -\frac{\lambda}{\sigma_z^3}\frac{df_{z_0}\left(\frac{x}{\sigma_z}\right)}{dx}\bigg|_{x=\lambda} - \frac{1}{\sigma_z^2}f_{z_0}\left(\frac{\lambda}{\sigma_z}\right) - \frac{\lambda}{\sigma_z^3}\frac{f_{z_0}^2(\lambda/\sigma_z)}{2[1 - F_{z_0}(\lambda/\sigma_z)]} \geqslant 0, \tag{19}$$

$$\Rightarrow \quad \frac{\sigma_z}{\lambda} \leqslant g_{\rm LO}^{z_0}\left(\frac{\lambda}{\sigma_z}\right) - \frac{f_{z_0}\left(\frac{\lambda}{\sigma_z}\right)}{2\left[1 - F_{z_0}\left(\frac{\lambda}{\sigma_z}\right)\right]}, \tag{20}$$

where the equality of Eq. (20) gives the non-zero solution $\sigma_z^{\rm opt}$. The numerical solution of $\sigma_z^{\rm opt}$ can refer to [33]. When the noise level $0 < \sigma_z < \sigma_z^{\rm opt}$, the derivative $\partial\xi_{\rm DZ}/\partial\sigma_z > 0$, and the noise-enhanced effect will appear in the dead-zone limiter detector of Eq. (11). □

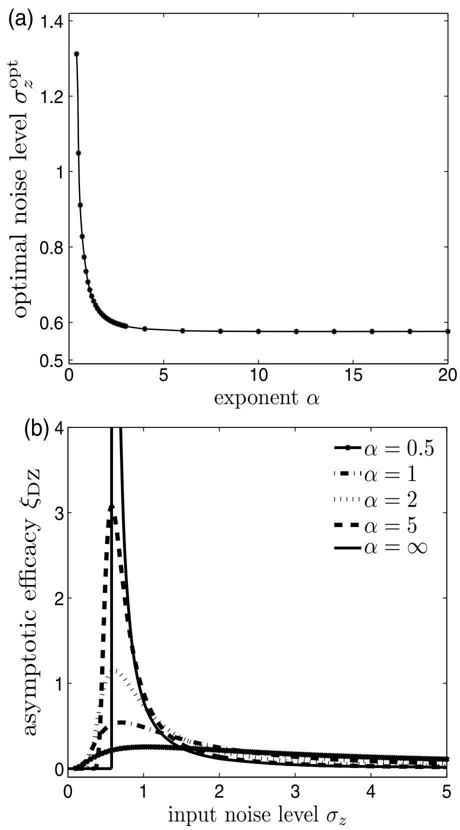

Fig. 1. (a) The optimal noise level σ_z^{opt} solved by Eq. (23) versus the exponent α in Eq. (21). (b) The normalized asymptotic efficacy ξ_{DZ} of Eq. (15) for the dead-zone limiter detector as a function of noise level σ_z for different exponents $\alpha = 0.5, 1, 2, 5$ and ∞ in Eq. (21). Here, the response threshold $\lambda = 1$ in Eq. (12).

Example 1. The non-Gaussian noise is often useful for modeling practical noisy environments where signals and systems are operated [31,32]. For example, a non-Gaussian model is the generalized Gaussian noise with PDF

$$f_z(x) = \frac{c_1}{\sigma_z}\exp\left(-c_2\left|\frac{x}{\sigma_z}\right|^{\alpha}\right), \tag{21}$$

where $c_1 = \frac{\alpha}{2}\Gamma^{\frac{1}{2}}(\frac{3}{\alpha})/\Gamma^{\frac{3}{2}}(\frac{1}{\alpha})$ and $c_2 = [\Gamma(\frac{3}{\alpha})/\Gamma(\frac{1}{\alpha})]^{\frac{\alpha}{2}}$. A positive exponent α allows us conveniently consider a spectrum of densities ranging from the Gaussian to those with relatively much faster or slower rates of exponential decay of their tails [31]. The corresponding nonlinearity of Eq. (14) is

$$g_{\mathrm{LO}}^{z_0}(x) = \alpha c_2 |x|^{\alpha-1}\,\mathrm{sign}(x). \tag{22}$$

For the dead-zone limiter detector of Eq. (11), Eq. (13) becomes

$$\frac{\sigma_z}{\lambda} = \alpha c_2\left|\frac{\lambda}{\sigma_z}\right|^{\alpha-1} - \frac{c_1\exp(-c_2|\frac{\lambda}{\sigma_z}|^{\alpha})}{2[1-F_{z_0}(\frac{\lambda}{\sigma_z})]}. \tag{23}$$

Without loss of generality, the response threshold takes $\lambda = 1$, and the optimal noise level σ_z^{opt} is shown in Fig. 1(a) as a function of the exponent α. It is illustrated in Fig. 1(b) that, as the noise level σ_z increases from zero to σ_z^{opt}, the normalized asymptotic efficacy ξ_{DZ} is enhanced to its maximum for different exponents $\alpha = 0.5, 1, 2, 5$ and ∞. Fig. 1(a) also shows that, as the exponent α increases, the optimal level of σ_z^{opt} tends to a constant value of $1/\sqrt{3}$, which is just the optimal noise level σ_z^{opt} corresponding to $\alpha = \infty$ (uniform noise), as shown in Fig. 1(b).

Corollary 3. *No noise-enhanced detection phenomenon will occur for the sign detector of Eq. (11) with threshold $\lambda = 0$ and characteristic $g_{\mathrm{DZ}}(x) = \mathrm{sign}(x)$.*

Proof. From Eq. (5), the normalized asymptotic efficacy ξ_{DZ} of the sign detector is

$$\xi_{\mathrm{DZ}} = \frac{\mathrm{E}_z^2[g'_{\mathrm{DZ}}(x)]}{\mathrm{E}_z[g_{\mathrm{DZ}}^2(x)]} = 4f_z^2(0) = \frac{4f_{z_0}^2(0)}{\sigma_z^2}. \tag{24}$$

Then, we find

$$\frac{\partial \xi_{\mathrm{DZ}}}{\partial \sigma_z} = -\frac{8f_{z_0}^2(0)}{\sigma_z^3} \leqslant 0, \tag{25}$$

for $\sigma_z > 0$. Therefore, no noise-enhanced detection phenomenon will occur. □

2.3. Noise enhancement by adding noise

The received signal is often corrupted by noise before it arrives at the detector. We now add additional noise to a given observation vector X in the context of SR. The updated components

$$\hat{X}_n = \theta s_n + Z_n + Y_n = \theta s_n + W_n, \tag{26}$$

where the added i.i.d. random variables Y_n are with PDF f_y and variance σ_y^2. Then, the composite components W_n have a convolved PDF $f_w(x) = \int_{-\infty}^{\infty} f_y(x-u) f_z(u)\,du$. In this case, the normalized asymptotic efficacy of Eq. (5) is updated as

$$\hat{\xi}_{\mathrm{GC}} = \frac{\mathrm{E}_w^2[g'(x)]}{\mathrm{E}_w[g^2(x)]} \leqslant \mathrm{E}_w\left[\frac{f_w'^2(x)}{f_w^2(x)}\right] = \hat{\xi}_{\mathrm{LO}} = I(f_w), \tag{27}$$

with the Fisher information $I(f_w)$ of f_w. Here, the equality is achieved by an updated locally optimal detector

$$\hat{T}_{\mathrm{LO}}(\hat{X}) = \sum_{n=1}^{N} \hat{g}_{\mathrm{LO}}(\hat{X}_n) s_n \underset{H_0}{\overset{H_1}{\gtrless}} \gamma, \tag{28}$$

based on the locally optimal nonlinearity

$$\hat{g}_{\mathrm{LO}}(x) = C\frac{f_w'(x)}{f_w(x)}. \tag{29}$$

Furthermore, we assume $f_w(x) = f_{w_0}(x/\sigma_w)/\sigma_w$, and f_{w_0} is the standardized noise PDF with unity variance. Then, we have the following corollaries.

Corollary 4. *No noise-enhanced detection phenomenon will occur in the updated locally optimal detector of Eq. (28).*

Proof. For the composite noise components W_n, the noise variance $\sigma_w^2 = \sigma_z^2 + \sigma_y^2$ and the initial noise variance σ_z^2 is fixed. Then, we have

$$\frac{\partial \hat{\xi}_{\mathrm{LO}}}{\partial \sigma_y} = \frac{\partial \hat{\xi}_{\mathrm{LO}}}{\partial \sigma_w}\frac{\partial \sigma_w}{\partial \sigma_y} = \frac{\partial I(f_w)}{\partial \sigma_w}\frac{\sigma_y}{\sqrt{\sigma_z^2+\sigma_y^2}}$$
$$= -\frac{2\sigma_y I(f_{w_0})}{\sigma_w^4} < 0, \tag{30}$$

where $I(f_{w_0}) > 0$ is the Fisher information of f_{w_0}. Then, Corollary 4 is deduced. □

Corollary 5. *When the noise level $0 < \sigma_y < \sigma_y^{\mathrm{opt}}$, the noise-enhanced detection phenomenon will occur for the dead-zone limiter detector of Eq. (11). Here, for a fixed noise level σ_z, the optimal noise level*

$$\sigma_y^{\mathrm{opt}} = \sqrt{\left(\sigma_w^{\mathrm{opt}}\right)^2 - \sigma_z^2}, \tag{31}$$

and σ_w^{opt} is the non-zero solution of

$$\frac{\sigma_w}{\lambda} = \hat{g}_{\mathrm{LO}}^{w_0}\left(\frac{\lambda}{\sigma_w}\right) - \frac{f_{w_0}\left(\frac{\lambda}{\sigma_w}\right)}{2\left[1 - F_{w_0}\left(\frac{\lambda}{\sigma_w}\right)\right]}, \tag{32}$$

with the nonlinearity

$$\hat{g}_{\mathrm{LO}}^{w_0}(x) = -\frac{f'_{w_0}(x)}{f_{w_0}(x)}. \tag{33}$$

Proof. For the composite noise components W_n, the normalized asymptotic efficacy of the dead-zone limiter detector of Eq. (11) can be calculated as

$$\hat{\xi}_{\mathrm{DZ}} = \frac{\mathrm{E}_w^2[g'_{\mathrm{DZ}}(x)]}{\mathrm{E}_w[g_{\mathrm{DZ}}^2(x)]} = \frac{2 f_w^2(\lambda)}{1 - F_w(\lambda)}, \tag{34}$$

where F_w represents the cumulative distribution function of W_n. Then, the noise-enhanced detection effect will occur as

$$\frac{\partial \hat{\xi}_{\mathrm{DZ}}}{\partial \sigma_y} = \frac{\partial \hat{\xi}_{\mathrm{DZ}}}{\partial \sigma_w}\frac{\partial \sigma_w}{\partial \sigma_y} = \frac{\partial \hat{\xi}_{\mathrm{DZ}}}{\partial \sigma_w}\frac{\sigma_y}{\sigma_w} \geqslant 0 \quad \Rightarrow \quad \frac{\partial \hat{\xi}_{\mathrm{DZ}}}{\partial \sigma_w} \geqslant 0. \tag{35}$$

The demonstration is similar to the proof of Corollary 2, and the occurrence condition is indicated by Eq. (32). Correspondingly, the optimal added noise level σ_y^{opt} and σ_w^{opt} can be solved by Eqs. (31) and (32). □

Example 2. Assume the initial Gaussian noise components Z_n are with PDF $f_z(x) = \frac{1}{\sqrt{2\pi\sigma_z^2}}\exp(-\frac{x^2}{2\sigma_z^2})$ and fixed variance σ_z^2. The added uniform random variables Y_n have PDF $f_y(x) = 1/(2b)$ for $-b \leqslant x \leqslant b$ and zero otherwise. The composite random variables W_n have PDF

$$f_w(x) = \frac{\mathrm{Q}\left(\frac{x-b}{\sigma_z}\right) - \mathrm{Q}\left(\frac{x+b}{\sigma_z}\right)}{2b}. \tag{36}$$

For the dead-zone limiter detector in Eq. (11), the normalized asymptotic efficacy of Eq. (34) can be expressed as

$$\hat{\xi}_{\mathrm{DZ}} = \frac{\left[\mathrm{Q}\left(\frac{\lambda-b}{\sigma_z}\right) - \mathrm{Q}\left(\frac{\lambda+b}{\sigma_z}\right)\right]^2}{\int_\lambda^\infty b\left[\mathrm{Q}\left(\frac{x-b}{\sigma_z}\right) - \mathrm{Q}\left(\frac{x+b}{\sigma_z}\right)\right]dx}. \tag{37}$$

Then, the noise-enhanced effect will occur for $\partial\hat{\xi}_{\mathrm{DZ}}/\partial b \geqslant 0$, this is

$$\begin{aligned}
&2\left[f_z(\lambda-b) + f_z(\lambda+b)\right]\int_\lambda^\infty b\left[\mathrm{Q}\left(\frac{x-b}{\sigma_z}\right) - \mathrm{Q}\left(\frac{x+b}{\sigma_z}\right)\right]dx \\
&\quad - \left[\mathrm{Q}\left(\frac{\lambda-b}{\sigma_z}\right) - \mathrm{Q}\left(\frac{\lambda+b}{\sigma_z}\right)\right] \\
&\quad \times \left\{\int_\lambda^\infty \left[\mathrm{Q}\left(\frac{x-b}{\sigma_z}\right) - \mathrm{Q}\left(\frac{x+b}{\sigma_z}\right)\right]dx \right. \\
&\quad \left. + b\left[f_z(x-b) + f_z(x+b)\right]dx\right\} \geqslant 0,
\end{aligned} \tag{38}$$

$$\begin{aligned}
\Rightarrow \quad & 4b^2\left[f_z(\lambda-b) + f_z(\lambda+b)\right]\int_\lambda^\infty f_w(x)\,dx \\
& - 2b^2 f_w(\lambda)\left\{2\int_\lambda^\infty f_w(x)\,dx \right. \\
& \left. + \left[\mathrm{Q}\left(\frac{\lambda-b}{\sigma_z}\right) + \mathrm{Q}\left(\frac{\lambda+b}{\sigma_z}\right)\right]\right\} \geqslant 0.
\end{aligned} \tag{39}$$

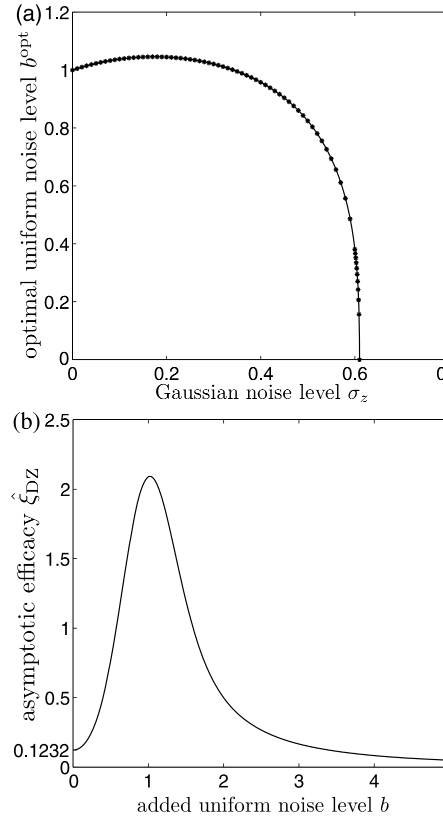

Fig. 2. (a) The optimal level b^{opt} of added uniform noise versus the initial Gaussian noise level σ_z. Here, the dead-zone detector is with the response threshold $\lambda = 1$. (b) The normalized asymptotic efficacy $\hat{\xi}_{\mathrm{DZ}}$ of Eq. (37) for the dead-zone limiter detector as a function of the added uniform noise level b. Here, the initial Gaussian noise level is fixed as $\sigma_z = 0.3$.

Thus, the optimal uniform noise level b^{opt} can be solved by

$$\begin{aligned}
& f_w(\lambda)\left[\mathrm{Q}\left(\frac{\lambda-b}{\sigma_z}\right) + \mathrm{Q}\left(\frac{\lambda+b}{\sigma_z}\right)\right] + 2 f_w(\lambda)\left[1 - F_w(\lambda)\right] \\
&\quad = 2\left[f_z(\lambda-b) + f_z(\lambda+b)\right]\left[1 - F_w(\lambda)\right],
\end{aligned} \tag{40}$$

and the noise-enhanced effect will occur as the uniform level $0 < b < b^{\mathrm{opt}}$. Without loss of generality, the response threshold takes $\lambda = 1$, and the optimal uniform noise level b^{opt} is plotted in Fig. 2(a) as a function of the initial Gaussian noise level σ_z. For instance, when the initial Gaussian noise level $\sigma_z = 0.3$, the corresponding normalized asymptotic efficacy of Eq. (15) is $\xi_{\mathrm{DZ}} = 0.1232$ without the addition of uniform noise ($b = 0$). When $b < b^{\mathrm{opt}} = 1.02$, the addition of uniform noise is helpful for weak signal detection, as shown in Fig. 2(b). We see that the normalized asymptotic efficacy can be improved up to $\hat{\xi}_{\mathrm{DZ}} = 2.092$ at $b^{\mathrm{opt}} = 1.02$, as illustrated in Fig. 2(b).

An important issue is that, for a given noise level σ_z, we can tune the threshold λ to maximize the normalized asymptotic efficacy ξ_{DZ} [23,29–32]. Michels et al. demonstrated that the normalized asymptotic efficacy of the tuned dead-zone limiter detector with optimal threshold λ^{opt} cannot be improved by adding noise to the signal [30] (Section 5.3, pp. 33–35). For the case of where the threshold is not optimal, they further proved that the optimal detection performance can be achieved by adding independent dichotomous noise [23]. For a fixed threshold λ, Corollaries 2 and 5 apply our general characterization to the dead-zone limiter detector for any type of scaled noise. The optimal noise level can be solved by Eq. (13) and Eq. (32).

For the scaled noise PDF $f_z(x) = f_{z_0}(x/\sigma_z)/\sigma_z$ with a given noise level σ_z and based on Eq. (15), the optimum threshold λ^{opt} can be solved by

$$\frac{\partial \xi_{\text{DZ}}}{\partial \lambda} = 0, \tag{41}$$

$$\Rightarrow \quad 4 f_z(\lambda) f_z'(\lambda)\bigl[1 - F_z(\lambda)\bigr] + 2 f_z^2(\lambda) f_z(\lambda) = 0, \tag{42}$$

$$\Rightarrow \quad 2 g_{\text{LO}}^{z_0}(\lambda/\sigma_z)\bigl[1 - F_{z_0}(\lambda/\sigma_z)\bigr] - f_{z_0}(\lambda/\sigma_z) = 0. \tag{43}$$

In Example 2, the initial Gaussian noise is with a given noise level σ_z, Eq. (43) yields the optimal threshold $\lambda^{\text{opt}} = 0.612\sigma_z$. Thus, the fixed threshold $\lambda = 1$ is optimal for the initial noise level $\sigma_z = 1.634$. It is shown in Fig. 2(a) that, for the fixed threshold $\lambda = 1$, the non-zero solution of added uniform noise level b^{opt} only exits for the initial Gaussian noise level $0 < \sigma_z < 0.61$. In other words, for the given initial noise level $\sigma_z > 0.61$ (including the optimal matching noise level $\sigma_z = 1.634$ for threshold $\lambda = 1$), no enhancement by noise can take place. In this respect, our results here accord with the conclusions of [29,30] that the normalized asymptotic efficacy of the dead-zone limiter detector with optimal threshold cannot be improved by adding noise, but the SR effect is possible when the threshold is not optimal for the initial given noise level.

2.4. Noise enhancement in a parallel array of nonlinearities

The constructive role of internal noise has been adequately reappraised for improving the performance of an array of nonlinearities [3,4,7–10]. Compared with an isolated nonlinearity, the performance of an array can be much improved by the internal noise [3,4,7–10]. Moreover, the positive role of noise does not need to occur for an isolated nonlinearity, but can come into play in a parallel array of nonlinearities [4,7–10].

Let $\hat{X}_m = (\hat{X}_{m1}, \hat{X}_{m2}, \ldots, \hat{X}_{mN})$ be the vector of N observation components at the m-th element of receiving array of M identical nonlinearities. In this observation model [4], $\hat{X}_{mn} = X_n + Y_{mn} = \theta s_n + Z_n + Y_{mn} = \theta s_n + W_{mn}$. Here, in each nonlinearity g, the M noise terms Y_m are assumed to be mutually independent with the same PDF f_y and variance σ_y^2. Then, at the observed time n, the array outputs are collected as $\bar{g}_n = \sum_{m=1}^{M} g(\hat{X}_{mn})/M$, and the generalized correlation detector can be constructed as

$$T_{\text{GC}}(\hat{X}) = \sum_{n=1}^{N} \bar{g}_n s_n \underset{H_0}{\overset{H_1}{\gtrless}} \gamma. \tag{44}$$

The statistic T_{GC} is also asymptotically Gaussian for a sufficiently large observed size N. Under the null hypothesis H_0, the mean $\mathrm{E}_w[T_{\text{GC}}|H_0] = \mathrm{E}_w[g(w)]\sum_{n=1}^{N} s_n = 0$ and the variance

$$\begin{aligned}
\operatorname{var}[T_{\text{GC}}|H_0] &= \mathrm{E}_w\bigl[T_{\text{GC}}^2\big|H_0\bigr] - \mathrm{E}_w^2[T_{\text{GC}}|H_0] \\
&= N P_s \mathrm{E}_z\left\{\frac{1}{M^2}\sum_{m=1}^{M}\sum_{k=1}^{M}\mathrm{E}_y\bigl[g(W_m)g(W_k)\bigr]\right\} \\
&= \frac{N P_s}{M^2}\mathrm{E}_z\bigl\{M\mathrm{E}_y\bigl[g^2(W_m)\bigr] \\
&\quad + M(M-1)\mathrm{E}_y\bigl[g(W_m)g(W_k)\bigr]\bigr\} \quad (\forall m \neq k) \\
&= \frac{N P_s}{M}\bigl\{\mathrm{E}_w\bigl[g^2(w)\bigr] \\
&\quad + (M-1)\mathrm{E}_z\bigl\{\mathrm{E}_y^2\bigl[g(y+z)\bigr]\bigr\}\bigr\},
\end{aligned} \tag{45}$$

where $\mathrm{E}_z\{\mathrm{E}_y[g(W_m)g(W_k)]\} = \mathrm{E}_z\{\mathrm{E}_y^2[g(w)]\} = \mathrm{E}_z\{\mathrm{E}_y^2[g(y+z)]\}$. Under the hypothesis H_1 and as the signal strength $\theta \to 0$, the mean has the asymptotic form

$$\begin{aligned}
\mathrm{E}_w[T_{\text{GC}}|H_1] &= \mathrm{E}_w\left[\sum_{n=1}^{N}\frac{1}{M}\sum_{m=1}^{M} g(\theta s_n + W_{mn}) s_n\right] \\
&\approx \mathrm{E}_w\left\{\sum_{n=1}^{N}\bigl[g(w) + \theta s_n g'(w)\bigr] s_n\right\} \\
&= \mathrm{E}_w\left[\sum_{n=1}^{N}\theta s_n^2 g'(w)\right] \\
&= \theta N P_s \mathrm{E}_w\bigl[g'(w)\bigr],
\end{aligned} \tag{46}$$

and variance $\operatorname{var}[T_{\text{GC}}|H_1] \approx \operatorname{var}[T_{\text{GC}}|H_0]$. Then, the normalized asymptotic efficacy of the detector in Eq. (44) is given by

$$\begin{aligned}
\hat{\xi}_{\text{GC}} &= \lim_{N\to\infty}\frac{\bigl\{\frac{d\mathrm{E}_w[T_{\text{GC}}(\hat{X})]}{d\theta}\big|_{\theta=0}\bigr\}^2}{N P_s \operatorname{var}[T_{\text{GC}}(\hat{X})]|_{\theta=0}} \\
&= \frac{\mathrm{E}_w^2[g'(w)]}{\frac{1}{M}\mathrm{E}_w[g^2(w)] + \frac{M-1}{M}\mathrm{E}_z\{\mathrm{E}_y^2[g(y+z)]\}}.
\end{aligned} \tag{47}$$

Example 3. We choose the characteristic $g(x) = \operatorname{sign}(x)$ in the detector of Eq. (44). The initial noise $Z(t)$ is Gaussian distributed, and the M array noise terms $Y_m(t)$ are uniformly random variables. The composite noise $W_m(t)$ are with the convolved PDF f_w of Eq. (36), as indicated in Example 2. Therefore, the normalized asymptotic efficacy is computed as

$$\begin{aligned}
\hat{\xi}_{\text{DZ}} &= \frac{4 f_w^2(0)}{\frac{1}{M}\mathrm{E}_w[\operatorname{sign}^2(w)] + \frac{M-1}{M}\mathrm{E}_z\{\mathrm{E}_y^2[\operatorname{sign}(z+y)]\}} \\
&= \frac{4 f_w^2(0)}{\frac{1}{M} + \frac{M-1}{M}\mathrm{E}_z\bigl[\frac{(|z+b|-|z-b|)^2}{4b^2}\bigr]}.
\end{aligned} \tag{48}$$

Since the noise-enhanced phenomenon occurs when $\partial\hat{\xi}_{\text{DZ}}/\partial b \geqslant 0$, it is found that the optimal noise level b^{opt} is the solution of

$$\begin{aligned}
&\bigl[f_w(0) - f_z(b)\bigr]\left\{1 + (M-1)\mathrm{E}_z\left[\frac{(|z+b|-|z-b|)^2}{4b^2}\right]\right\} \\
&\quad = (M-1) f_w(0)\mathrm{E}_z\left\{\frac{(|z+b|-|z-b|)^2}{2b^2}\right. \\
&\qquad \left. - \frac{(|z+b|-|z-b|)[\operatorname{sign}(z+b) + \operatorname{sign}(z-b)]}{2b}\right\}.
\end{aligned} \tag{49}$$

For the array size $M = 1$, Eq. (49) yields $f_w(0) - f_z(b) = 0$ and the optimal uniform noise level $b^{\text{opt}} = 0$. Thus, there is no noise-enhanced effect in the detector of Eq. (44) with a single nonlinearity. For a fixed Gaussian noise level $\sigma_z = 0.3$, the optimal added uniform noise level b^{opt} is illustrated as a function of the array size M in Fig. 3(a). It is shown in Fig. 3(b) that the normalized asymptotic efficacy $\hat{\xi}_{\text{DZ}}$ varies as a function of added uniform noise level b for different array sizes. For a single nonlinearity g, it is seen that the added uniform noise is no use for the performance enhancement of the detector ($M = 1$). As $M \geqslant 2$, it is seen in Fig. 3(b) that the added uniform noise can enhance the normalized asymptotic efficacy $\hat{\xi}_{\text{DZ}}$, and the noise-enhanced effect does occur. Moreover, as the array size M increases, the peak value of $\hat{\xi}_{\text{DZ}}$ is also improved gradually by tuning the added uniform noise level into the corresponding optimal value of b^{opt}, as shown in Fig. 3(b).

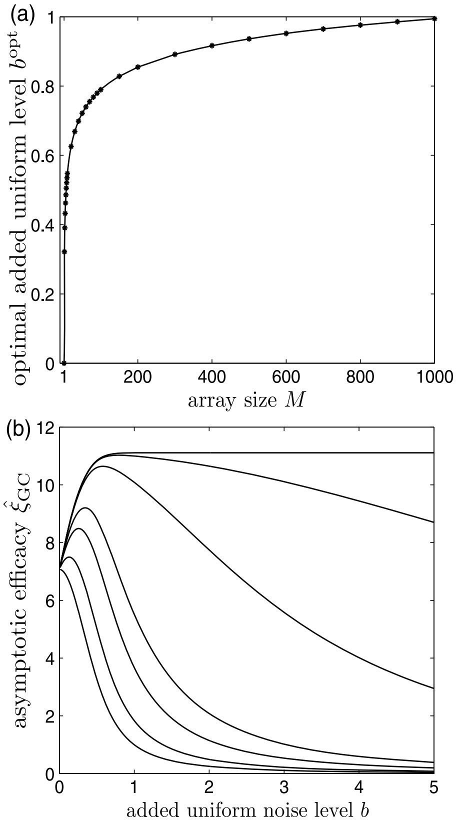

Fig. 3. (a) The optimal level $b^{\rm opt}$ of the added uniform noise versus the array size M for the detector of Eq. (44). (b) The normalized asymptotic efficacy $\hat{\xi}_{\rm DZ}$ as a function of the added uniform noise level b and the array size M. From the bottom upwards, $M = 1, 2, 5, 10, 100, 1000, \infty$. Here, the initial Gaussian noise level $\sigma_z = 0.3$ and the nonlinearity $g(x) = \text{sign}(x)$.

3. Conclusion

In this paper, we study the noise-enhanced detection of a weak known signal in additive white noise. For a sufficiently large observation size, the performance of a generalized correlation detector is determined by the normalized asymptotic efficacy $\xi_{\rm GC}$. Then, the positive derivative of $\xi_{\rm GC}$ with respect to the noise level indicates the occurrence of the noise-enhanced detection effect. According to this condition, we arrive at some interesting conclusions on whether the role of noise in a generalized correlation detector offers an enhancement or not.

We here only consider some analytical nonlinearities, e.g. the dead-zone limiter nonlinearity and the locally optimal nonlinearity. There are other interesting nonlinearities such as the saturation nonlinearity [34] and the soft-threshold nonlinearity [35], which can be of interest for further studies of weak signal detection in the context of SR.

Acknowledgment

This work is sponsored by the NSF of Shandong Province (No. ZR2010FM006).

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