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Quantum information, quantum computation : An introduction.

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Motivations pour le quantique

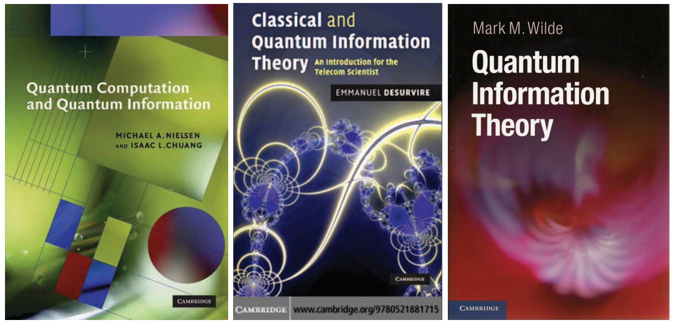
pour le traitement de l'information :

1) Quand on utilise des systèmes élémentaires (photons, électrons, atomes, nanodevices, ...).

2) Pour bénéficier d'effets purement quantiques (parallèlisme, intrication, ...).

3) Domaine de recherche récent, riche et largement ouvert.

Some recent textbooks



M. Nielsen & I. Chuang 2000, 676 pages

E. Desurvire 2009, 691 pages

M. Wilde 2013, 655 pages

arXiv:1106.1445v5 [quant-ph] M. Wilde, "From classical to quantum Shannon theory", 670 pages.

Quantum system

Represented by a state vector $|\psi\rangle$
in a complex Hilbert space $\mathcal{H}$,
with unit norm $\langle\psi|\psi\rangle = \|\psi\|^2 = 1$.

In dimension 2 : the qubit (photon, electron, atom, ...)

State $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$
in some orthonormal basis $\{|0\rangle, |1\rangle\}$ of $\mathcal{H}_2$,
with complex $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$.

$|\psi\rangle = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$, $\quad |\psi\rangle^\dagger = \langle\psi| = [\alpha^*, \beta^*] \quad \Longrightarrow \langle\psi|\psi\rangle = \|\psi\|^2 = |\alpha|^2 + |\beta|^2$ scalar.

$|\psi\rangle\langle\psi| = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}[\alpha^*, \beta^*] = \begin{bmatrix}\alpha\alpha^* & \alpha\beta^*\\ \alpha^*\beta & \beta\beta^*\end{bmatrix} = \Pi_\psi$ orthogonal projector on $|\psi\rangle$.

Measurement of the qubit

When a qubit in state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$
is measured in the orthonormal basis $\{|0\rangle, |1\rangle\}$,

$\Longrightarrow$ only 2 possible outcomes (Born rule) :
state $|0\rangle$ with probability $|\alpha|^2 = |\langle 0|\psi\rangle|^2$, or
state $|1\rangle$ with probability $|\beta|^2 = |\langle 1|\psi\rangle|^2$.

Measurement :
- a probabilistic process,
- as a projection of the state $|\psi\rangle$ in an orthonormal basis,
- with statistics evaluable over repeated experiments with same preparation $|\psi\rangle$.

Hadamard basis

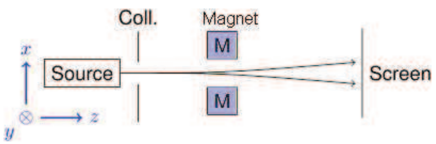
Another orthonormal basis of $\mathcal{H}_2$

$$\left\{ |+\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big); \quad |-\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle - |1\rangle\Big) \right\}.$$

$\Longleftrightarrow$ Computational orthonormal basis

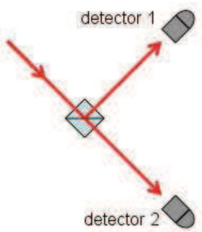
$$\left\{ |0\rangle = \frac{1}{\sqrt{2}}\Big(|+\rangle + |-\rangle\Big); \quad |1\rangle = \frac{1}{\sqrt{2}}\Big(|+\rangle - |-\rangle\Big) \right\}.$$

Experiments



Stern-Gerlach apparatus for particles with two states of spin (electron, atom).

Two states of polarization of a photon :
(Nicol prism, Glan-Thompson, ...)

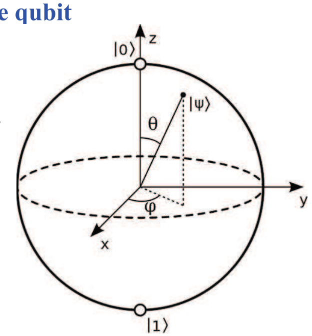


Bloch sphere representation of the qubit

Qubit in state
$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$.

$\Longleftrightarrow |\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\varphi}\sin(\theta/2)\,|1\rangle$

with $\theta \in [0, \pi]$,
$\varphi \in [0, 2\pi[$.



As a quantum object
the qubit has infinitely many degrees of freedom (θ, φ),
yet when it is measured it can only be found in one of two states
(just like a classical bit).

In dimension N (finite) (extensible to infinite dimension)

State $|\psi\rangle = \sum_{n=1}^{N} \alpha_n |n\rangle$, in some orthonormal basis $\{|1\rangle, |2\rangle, \dots |N\rangle\}$ of $\mathcal{H}_N$,

with $\alpha_n \in \mathbb{C}$, and $\sum_{n=1}^{N} |\alpha_n|^2 = \langle\psi|\psi\rangle = 1$.

Proba. $\Pr\{|n\rangle\} = |\alpha_n|^2$ in a projective measurement of $|\psi\rangle$ in basis $\{|n\rangle\}$.

Inner product $\langle k|\psi\rangle = \sum_{n=1}^{N} \alpha_n \overbrace{\langle k|n\rangle}^{\delta_{kn}} = \alpha_k$ coordinate.

$\mathrm{S} = \sum_{n=1}^{N} |n\rangle\langle n| = \mathrm{I}_N$ identity of $\mathcal{H}_N$ (closure or completeness relation),

since, $\forall\,|\psi\rangle : \mathrm{S}\,|\psi\rangle = \sum_{n=1}^{N} |n\rangle \overbrace{\langle n|\psi\rangle}^{\alpha_n} = \sum_{n=1}^{N} \alpha_n |n\rangle = |\psi\rangle \Longrightarrow \mathrm{S} = \mathrm{I}_N$.

Multiple qubits

A system (a word) of N qubits has a state in $\mathcal{H}_2^{\otimes N}$,

a tensor-product vector space with dimension 2^N,

and orthonormal basis $\{|x_1 x_2 \cdots x_N\rangle\}_{\vec{x} \in \{0,1\}^N}$.

Example $N = 2$:

Generally $|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$.

Or, as a special separable state

$$|\phi\rangle = \Big(\alpha_1|0\rangle + \beta_1|1\rangle\Big) \otimes \Big(\alpha_2|0\rangle + \beta_2|1\rangle\Big)$$
$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle .$$

A multipartite state which is not separable is entangled.

Entangled states

- Example of a separable state of two qubits AB :

$$|AB\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big) \otimes \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big) = \frac{1}{2}\Big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\Big).$$

When measured in the basis $\{|0\rangle, |1\rangle\}$, each qubit A and B can be found in state $|0\rangle$ or $|1\rangle$ independently with probability $1/2$.

$$\Pr\{|A\rangle = |0\rangle\} = \Pr\{|AB\rangle = |00\rangle\} + \Pr\{|AB\rangle = |01\rangle\} = 1/4 + 1/4 = 1/2.$$

- Example of an entangled state of two qubits AB :

$$|AB\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big). \qquad \Pr\{|A\rangle = |0\rangle\} = \Pr\{|AB\rangle = |00\rangle\} = 1/2.$$

When measured in the basis $\{|0\rangle, |1\rangle\}$, each qubit A and B can be found in state $|0\rangle$ or $|1\rangle$ with probability $1/2$ (randomly, no predetermination before measurement).
But if A is found in $|0\rangle$ necessarily B is found in $|0\rangle$,
and if A is found in $|1\rangle$ necessarily B is found in $|1\rangle$,
no matter how distant the two qubits are before measurement.

Bell basis

A pair of qubits in $\mathcal{H}_2^{\otimes 2}$ is a quantum system with dimension 4,

with original (computational) orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

Another useful orthonormal basis of $\mathcal{H}_2^{\otimes 2}$ is the Bell basis $\{|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle\}$,

with
$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$$
$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big)$$
$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big)$$
$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big).$$

Observables

For a quantum system in $\mathcal{H}_N$ with dimension N,
a projective measurement is defined by an orthonormal basis $\{|1\rangle, \dots |N\rangle\}$ of $\mathcal{H}_N$,
and the N orthogonal projectors $|n\rangle\langle n|$, for $n = 1$ to N.

Also, any Hermitian (i.e. $\Omega = \Omega^\dagger$) operator Ω on $\mathcal{H}_N$,
has its eigenstates forming an orthonormal basis $\{|\omega_1\rangle, \dots |\omega_N\rangle\}$ of $\mathcal{H}_N$.
Therefore, any Hermitian operator Ω on $\mathcal{H}_N$ defines a valid measurement,
and has a spectral decomposition $\Omega = \sum_{n=1}^{N} \omega_n |\omega_n\rangle\langle\omega_n|$, with the real eigenvalues ω_n.

Also, any physical quantity measurable on a quantum system is represented in quantum theory by a Hermitian operator (an observable) Ω.

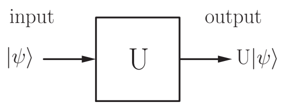
When system in state $|\psi\rangle$, measuring observable Ω is equivalent to performing a projective measurement in eigenbasis $\{|\omega_n\rangle\}$, with projectors $|\omega_n\rangle\langle\omega_n| = \Pi_n$, and yields the eigenvalue ω_n with probability $\Pr\{\omega_n\} = |\langle\omega_n|\psi\rangle|^2 = \langle\psi|\omega_n\rangle\langle\omega_n|\psi\rangle = \langle\psi|\Pi_n|\psi\rangle$.

Computation on a qubit

Through a unitary operator U on $\mathcal{H}_2$ (a 2×2 matrix) : (i.e. $\mathrm{U}^{-1} = \mathrm{U}^\dagger$)

normalized vector $|\psi\rangle \in \mathcal{H}_2 \longrightarrow \mathrm{U}|\psi\rangle$ normalized vector $\in \mathcal{H}_2$.

$\equiv$ quantum gate



Hadamard gate $\mathrm{H} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$. Identity gate $\mathrm{I}_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

$\mathrm{H}^2 = \mathrm{I}_2 \Longleftrightarrow \mathrm{H}^{-1} = \mathrm{H} = \mathrm{H}^\dagger$ Hermitian unitary.

$\mathrm{H}|0\rangle = |+\rangle$ and $\mathrm{H}|1\rangle = |-\rangle$

$\Longrightarrow$ in a compact notation $\mathrm{H}|x\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + (-1)^x|1\rangle\Big)$, $\forall x \in \{0, 1\}$.

Pauli gates

$\mathrm{X} = \sigma_x = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $\mathrm{Y} = \sigma_y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$, $\mathrm{Z} = \sigma_z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.

$\mathrm{X}^2 = \mathrm{Y}^2 = \mathrm{Z}^2 = \mathrm{I}_2$ Hermitian unitary. $\mathrm{XY} = i\mathrm{Z}$, etc ...

$\{\mathrm{I}_2, \mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$ a basis for operators on $\mathcal{H}_2$.

Hadamard gate $\mathrm{H} = \frac{1}{\sqrt{2}}(\mathrm{X} + \mathrm{Z})$.

$\mathrm{X} = \sigma_x$ the inversion or Not quantum gate. $\mathrm{X}|0\rangle = |1\rangle$, $\mathrm{X}|1\rangle = |0\rangle$.

$\mathrm{W} = \sqrt{\mathrm{X}} = \sqrt{\sigma_x} = \frac{1}{2}\begin{bmatrix}1+i & 1-i\\ 1-i & 1+i\end{bmatrix}$, such that $\mathrm{W}^2 = \mathrm{X}$,

is the square-root of Not, a typically quantum gate (no classical equivalent).

In general, the gates U and $e^{i\phi}\mathrm{U}$ give the same measurement statistics at the output, and are thus physically equivalent, in this respect.

Any single-qubit gate can always be expressed as $e^{i\phi}\mathrm{U}_\xi$ with

$$\mathrm{U}_\xi = \exp\left(-i\frac{\xi}{2}\vec{n}\,\vec{\sigma}\right) = \cos\left(\frac{\xi}{2}\right)\mathrm{I}_2 - i\sin\left(\frac{\xi}{2}\right)\vec{n}\,\vec{\sigma},$$

where $\vec{n} = [n_x, n_y, n_z]^\top$ is a real unit vector of $\mathbb{R}^3$,

and a "vector" of 2×2 matrices $\vec{\sigma} = [\sigma_x, \sigma_y, \sigma_z]$,

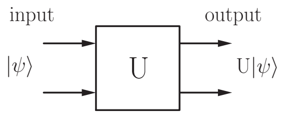
implementing in the Bloch sphere representation
a rotation of the qubit state of an angle ξ around the axis $\vec{n}$ in $\mathbb{R}^3$.

Computation on a pair of qubits

Through a unitary operator U on $\mathcal{H}_2^{\otimes 2}$ (a 4×4 matrix) :

normalized vector $|\psi\rangle \in \mathcal{H}_2^{\otimes 2} \longrightarrow \mathrm{U}|\psi\rangle$ normalized vector $\in \mathcal{H}_2^{\otimes 2}$.

$\equiv$ quantum gate
(always reversible)



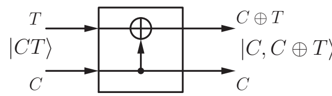
Completely defined for instance by the transformation of the four state vectors of the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

- Example : Controlled-Not gate

Via the XOR binary function : $a \oplus b = a$ when $b = 0$, or $= \overline{a}$ when $b = 1$;
invertible $a \oplus x = b \Longleftrightarrow x = a \oplus b = b \oplus a$.

Used to construct a unitary invertible quantum C-Not gate :
(T target, C control)

$|CT\rangle \longrightarrow |C, C \oplus T\rangle$
$|00\rangle \longrightarrow |00\rangle$
$|01\rangle \longrightarrow |01\rangle$
$|10\rangle \longrightarrow |11\rangle$
$|11\rangle \longrightarrow |10\rangle$



$$\mathrm{U} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$$

$(\text{C-Not})^2 = \mathrm{I}_2 \Longleftrightarrow (\text{C-Not})^{-1} = \text{C-Not} = (\text{C-Not})^\dagger$ Hermitian unitary.

Computation on a system of N qubits

Through a unitary operator U on $\mathcal{H}_2^{\otimes N}$ (a $2^N \times 2^N$ matrix) :

normalized vector $|\psi\rangle \in \mathcal{H}_2^{\otimes N} \longrightarrow \mathrm{U}\,|\psi\rangle$ normalized vector $\in \mathcal{H}_2^{\otimes N}$.

$\equiv$ quantum gate : N input qubits $\xrightarrow{\mathrm{U}}$ N output qubits.

Completely defined for instance by the transformation of the 2^N state vectors of the computational basis.

Any N-qubit quantum gate may always be composed from two-qubit C-Not gates and single-qubit gates (universality).

This forms the grounding of quantum computation.

No cloning theorem (1982)

¿ Possibility of a circuit (a unitary U) that would take any state $|\psi\rangle$, associated to an auxiliary register $|s\rangle$, to transform the input $|\psi\rangle\,|s\rangle$ into the cloned output $|\psi\rangle\,|\psi\rangle$?

$|\psi_1\rangle\,|s\rangle \xrightarrow{\mathrm{U}} \mathrm{U}(|\psi_1\rangle\,|s\rangle) = |\psi_1\rangle\,|\psi_1\rangle$ (would be).

$|\psi_2\rangle\,|s\rangle \xrightarrow{\mathrm{U}} \mathrm{U}(|\psi_2\rangle\,|s\rangle) = |\psi_2\rangle\,|\psi_2\rangle$ (would be).

Linear superposition $|\psi\rangle = \alpha_1\,|\psi_1\rangle + \alpha_2\,|\psi_2\rangle$

$$|\psi\rangle\,|s\rangle \xrightarrow{\mathrm{U}} \mathrm{U}(|\psi\rangle\,|s\rangle) = \mathrm{U}(\alpha_1\,|\psi_1\rangle\,|s\rangle + \alpha_2\,|\psi_2\rangle\,|s\rangle)$$
$$= \alpha_1\,|\psi_1\rangle\,|\psi_1\rangle + \alpha_2\,|\psi_2\rangle\,|\psi_2\rangle \qquad \text{since U linear.}$$

But $|\psi\rangle\,|\psi\rangle = |\psi\rangle \otimes |\psi\rangle = \big(\alpha_1\,|\psi_1\rangle + \alpha_2\,|\psi_2\rangle\big)\big(\alpha_1\,|\psi_1\rangle + \alpha_2\,|\psi_2\rangle\big)$

$$= \alpha_1^2\,|\psi_1\rangle\,|\psi_1\rangle + \alpha_1\alpha_2\,|\psi_1\rangle\,|\psi_2\rangle + \alpha_1\alpha_2\,|\psi_2\rangle\,|\psi_1\rangle + \alpha_2^2\,|\psi_2\rangle\,|\psi_2\rangle$$

$\neq \mathrm{U}(|\psi\rangle\,|s\rangle)$ in general. $\Longrightarrow$ No cloning U possible.

Quantum parallelism

For a system of N qubits,
a quantum gate is any unitary operator U from $\mathcal{H}_2^{\otimes N}$ onto $\mathcal{H}_2^{\otimes N}$.

The quantum gate U is completely defined
by its action on the 2^N basis states of $\mathcal{H}_2^{\otimes N}$: $\{|\vec{x}\rangle\,, \vec{x} \in \{0,1\}^N\}$,
just like a classical gate.

Yet, the quantum gate U can be operated
on any linear superposition of the basis states $\{|\vec{x}\rangle\,, \vec{x} \in \{0,1\}^N\}$.

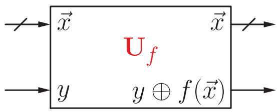
This is quantum parallelism, with no classical analog.

Parallel evaluation of a function (1/3)

A classical function $f(\cdot)$ from N bits to 1 bit

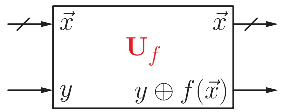
$$\vec{x} \in \{0,1\}^N \longrightarrow f(\vec{x}) \in \{0,1\}.$$

Used to construct a unitary operator U_f as an invertible f-controlled gate :



with binary output $y \oplus f(\vec{x}) = f(\vec{x})$ when $y = 0$, or $= \overline{f(\vec{x})}$ when $y = 1$.

Parallel evaluation of a function (2/3)



For every basis state $|\vec{x}\rangle$, with $\vec{x} \in \{0,1\}^N$:

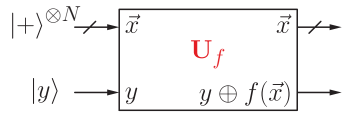
$$|\vec{x}\rangle\,|y=0\rangle \xrightarrow{\mathrm{U}_f} |\vec{x}\rangle\,|f(\vec{x})\rangle$$

$$|\vec{x}\rangle\,|y=1\rangle \longrightarrow |\vec{x}\rangle\,\big|\overline{f(\vec{x})}\big\rangle$$

$$|\vec{x}\rangle\,|+\rangle \longrightarrow |\vec{x}\rangle\,\frac{1}{\sqrt{2}}\Big[|f(\vec{x})\rangle + \big|\overline{f(\vec{x})}\big\rangle\Big] = |\vec{x}\rangle\,|+\rangle$$

$$|\vec{x}\rangle\,|-\rangle \longrightarrow |\vec{x}\rangle\,\frac{1}{\sqrt{2}}\Big[|f(\vec{x})\rangle - \big|\overline{f(\vec{x})}\big\rangle\Big] = |\vec{x}\rangle\,|-\rangle\,(-1)^{f(\vec{x})}$$

Parallel evaluation of a function (3/3)



$$|+\rangle^{\otimes N} = \left(\frac{1}{\sqrt{2}}\right)^N \sum_{\vec{x}\in\{0,1\}^N} |\vec{x}\rangle \quad \text{superposition of all basis states,}$$

$$|+\rangle^{\otimes N} \otimes |0\rangle \xrightarrow{\mathrm{U}_f} \left(\frac{1}{\sqrt{2}}\right)^N \sum_{\vec{x}\in\{0,1\}^N} |\vec{x}\,\rangle\,|f(\vec{x})\,\rangle \quad \text{superpos. of all values } f(\vec{x}).$$

$$|+\rangle^{\otimes N} \otimes |-\rangle \xrightarrow{\mathrm{U}_f} \left(\frac{1}{\sqrt{2}}\right)^N \sum_{\vec{x}\in\{0,1\}^N} |\vec{x}\,\rangle\,|-\rangle\,(-1)^{f(\vec{x})}$$

¿ How to extract, to measure, useful informations from superpositions ?

Deutsch-Jozsa algorithm (1992) : Parallel test of a function (1/5)

A classical function $f(\cdot)\left|\begin{array}{ccc} \{0,1\}^N & \longrightarrow & \{0,1\} \\ 2^N \text{ values} & \longrightarrow & 2 \text{ values,} \end{array}\right.$

can be constant or balanced (equal numbers of 0, 1 in output).

Classically : Between 2 and $\dfrac{2^N}{2} + 1$ evaluations of $f(\cdot)$ to decide.

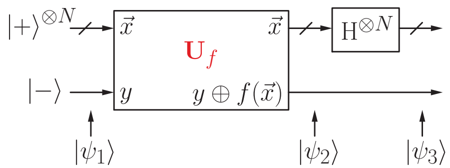
Quantumly : One evaluation of $f(\cdot)$ is enough (on a suitable superposition).

Lemma 1 : $\mathrm{H}\,|x\rangle = \dfrac{1}{\sqrt{2}}\Big(|0\rangle + (-1)^x\,|1\rangle\Big) = \dfrac{1}{\sqrt{2}} \displaystyle\sum_{z\in\{0,1\}} (-1)^{xz}\,|z\rangle\,, \quad \forall x \in \{0,1\}$

$$\Longrightarrow \mathrm{H}^{\otimes N}\,|\vec{x}\rangle = \mathrm{H}\,|x_1\rangle \otimes \cdots \otimes \mathrm{H}\,|x_N\rangle = \left(\frac{1}{\sqrt{2}}\right)^N \sum_{\vec{z}\in\{0,1\}^N} (-1)^{\vec{x}\vec{z}}\,|\vec{z}\rangle\,, \quad \forall \vec{x} \in \{0,1\}^N$$

with scalar product $\vec{x}\vec{z} = x_1 z_1 + \cdots + x_N z_N$ modulo 2. (quant. Hadamard transfo.)

Deutsch-Jozsa algorithm (2/5)



Input state $|\psi_1\rangle = |+\rangle^{\otimes N}\,|-\rangle = \left(\dfrac{1}{\sqrt{2}}\right)^N \displaystyle\sum_{\vec{x}\in\{0,1\}^N} |\vec{x}\,\rangle\,|-\rangle$

Internal state $|\psi_2\rangle = \left(\dfrac{1}{\sqrt{2}}\right)^N \displaystyle\sum_{\vec{x}\in\{0,1\}^N} |\vec{x}\,\rangle\,|-\rangle\,(-1)^{f(\vec{x})}$

Deutsch-Jozsa algorithm (3/5)

Output state $|\psi_3\rangle = \big(\mathrm{H}^{\otimes N} \otimes \mathrm{I}_2\big)\,|\psi_2\rangle$

$$= \left(\frac{1}{\sqrt{2}}\right)^N \sum_{\vec{x}\in\{0,1\}^N} \mathrm{H}^{\otimes N}\,|\vec{x}\,\rangle\,|-\rangle\,(-1)^{f(\vec{x})}$$

$$= \left(\frac{1}{2}\right)^N \sum_{\vec{x}\in\{0,1\}^N} \sum_{\vec{z}\in\{0,1\}^N} (-1)^{\vec{x}\vec{z}}\,|\vec{z}\,\rangle\,|-\rangle\,(-1)^{f(\vec{x})} \quad \text{by Lemma 1,}$$

or $|\psi_3\rangle = |\psi\rangle\,|-\rangle$, with $|\psi\rangle = \left(\dfrac{1}{2}\right)^N \displaystyle\sum_{\vec{z}\in\{0,1\}^N} u(\vec{z}\,)\,|\vec{z}\,\rangle$

and the scalar weight $u(\vec{z}\,) = \displaystyle\sum_{\vec{x}\in\{0,1\}^N} (-1)^{f(\vec{x}) + \vec{x}\vec{z}}$

Deutsch-Jozsa algorithm (4/5)

So $|\psi\rangle = \dfrac{1}{2^N} \sum_{\vec{z}\in\{0,1\}^N} u(\vec{z})\,|\vec{z}\,\rangle$ with $u(\vec{z}) = \sum_{\vec{x}\in\{0,1\}^N} (-1)^{f(\vec{x})+\vec{x}\vec{z}}$.

For $|\vec{z}\,\rangle = |\vec{0}\,\rangle = |0\rangle^{\otimes N}$ then $u(\vec{z}=\vec{0}\,) = \sum_{\vec{x}\in\{0,1\}^N} (-1)^{f(\vec{x})}$.

• When $f(\cdot)$ constant : $u(\vec{z}=\vec{0}\,) = 2^N(-1)^{f(\vec{0})} = \pm 2^N \Longrightarrow$ in $|\psi\rangle$ the amplitude of $|\vec{0}\,\rangle$ is ± 1, and since $|\psi\rangle$ is with unit norm $\Longrightarrow |\psi\rangle = \pm|\vec{0}\,\rangle$, and all other $u(\vec{z}\neq\vec{0}\,) = 0$.
$\Longrightarrow$ When $|\psi\rangle$ is measured, N states $|0\rangle$ are found.

• When $f(\cdot)$ balanced : $u(\vec{z}=\vec{0}\,) = 0 \Longrightarrow |\psi\rangle$ is not or does not contain state $|\vec{0}\,\rangle$.
$\Longrightarrow$ When $|\psi\rangle$ is measured, at least one state $|1\rangle$ is found.

$\longrightarrow$ Illustrates quantum ressources of parallelism, coherent superposition, interference.
(When $f(\cdot)$ is neither constant nor balanced $|\psi\rangle$ contains a little bit of $|\vec{0}\,\rangle$.)

Deutsch-Jozsa algorithm (5/5)

[1] D. Deutsch; "Quantum theory, the Church-Turing principle and the universal quantum computer"; *Proceedings of the Royal Society of London A* 400 (1985) 97–117.
The case $N = 2$.

[2] D. Deutsch, R. Jozsa; "Rapid solution of problems by quantum computation"; *Proceedings of the Royal Society of London A*, 439 (1993) 553–558.
Extension to arbitrary $N \geq 2$.

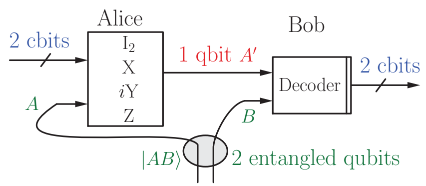
[3] E. Bernstein, U. Vazirani; "Quantum complexity theory"; *SIAM Journal on Computing* 26 (1997) 1411–1473.
Extension to $f(\vec{x}) = \vec{a}\vec{x}$ or $f(\vec{x}) = \vec{a}\vec{x} \oplus \vec{b}$, to find binary N-word $\vec{a} \longrightarrow$ by producing output $|\psi\rangle = |\vec{a}\,\rangle$.

[4] R. Cleve, A. Ekert, C. Macchiavello, M. Mosca; "Quantum algorithms revisited"; *Proceedings of the Royal Society of London A*, 454 (1998) 339–354.

Superdense coding (Bennett 1992) : exploiting entanglement

Alice and Bob share a qubit pair in entangled state $|AB\rangle = \dfrac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = |\beta_{00}\rangle$.

Alice chooses two classical bits, used to encode by applying to her qubit A one of $\{\mathrm{I}_2, \mathrm{X}, i\mathrm{Y}, \mathrm{Z}\}$, delivering the qubit A' sent to Bob.



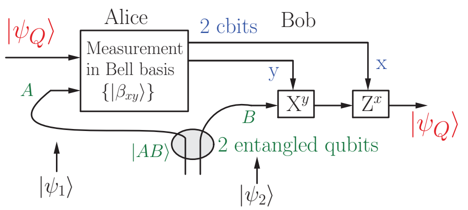
$\mathrm{I}_2 \otimes \mathrm{I}_2\,|AB\rangle = |\beta_{00}\rangle$
$\mathrm{X} \otimes \mathrm{I}_2\,|AB\rangle = |\beta_{01}\rangle$
$\mathrm{Z} \otimes \mathrm{I}_2\,|AB\rangle = |\beta_{10}\rangle$
$i\mathrm{Y} \otimes \mathrm{I}_2\,|AB\rangle = |\beta_{11}\rangle$

Bob receives this qubit A'. For decoding, Bob measures $|A'B\rangle$ in the Bell basis $\{|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle\}$, from which he recovers the two classical bits.

Teleportation (Bennett 1993) : of an unknown qubit state (1/3)

Qubit Q in unknown arbitrary state $|\psi_Q\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$.

Alice and Bob share a qubit pair in entangled state $|AB\rangle = \dfrac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = |\beta_{00}\rangle$.



Alice measures the pair of qubits QA in the Bell basis (so $|\psi_Q\rangle$ is locally destroyed), and the two resulting cbits x, y are sent to Bob.
Bob on his qubit B applies the gates X^y and Z^x which reconstructs $|\psi_Q\rangle$.

Teleportation (2/3)

$$|\psi_1\rangle = |\psi_Q\rangle|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\Big[\alpha_0|0\rangle\Big(|00\rangle + |11\rangle\Big) + \alpha_1|1\rangle\Big(|00\rangle + |11\rangle\Big)\Big]$$
$$= \frac{1}{\sqrt{2}}\Big[\alpha_0|000\rangle + \alpha_0|011\rangle + \alpha_1|100\rangle + \alpha_1|111\rangle\Big],$$

factorizable as
$$|\psi_1\rangle = \frac{1}{2}\Big[\frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)\Big(\alpha_0|0\rangle + \alpha_1|1\rangle\Big) + \frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big)\Big(\alpha_0|1\rangle + \alpha_1|0\rangle\Big) + \frac{1}{\sqrt{2}}\Big(|00\rangle - |11\rangle\Big)\Big(\alpha_0|0\rangle - \alpha_1|1\rangle\Big) + \frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big)\Big(\alpha_0|1\rangle - \alpha_1|0\rangle\Big)\Big],$$

Teleportation (3/3)

$$|\psi_1\rangle = \frac{1}{2}\Big[|\beta_{00}\rangle\Big(\alpha_0|0\rangle + \alpha_1|1\rangle\Big) + |\beta_{01}\rangle\Big(\alpha_0|1\rangle + \alpha_1|0\rangle\Big) + |\beta_{10}\rangle\Big(\alpha_0|0\rangle - \alpha_1|1\rangle\Big) + |\beta_{11}\rangle\Big(\alpha_0|1\rangle - \alpha_1|0\rangle\Big)\Big]$$

The first two qubits QA measured in Bell basis $\{|\beta_{xy}\rangle\}$ yield the two cbits xy, used to transform the third qubit B by X^y then Z^x, which reconstructs $|\psi_Q\rangle$.

When QA is measured in $|\beta_{00}\rangle$ then B is in $\alpha_0|0\rangle + \alpha_1|1\rangle \xrightarrow{\mathrm{I}_2} \cdot \xrightarrow{\mathrm{I}_2} |\psi_Q\rangle$

When QA is measured in $|\beta_{01}\rangle$ then B is in $\alpha_0|1\rangle + \alpha_1|0\rangle \xrightarrow{\mathrm{X}} \cdot \xrightarrow{\mathrm{I}_2} |\psi_Q\rangle$

When QA is measured in $|\beta_{10}\rangle$ then B is in $\alpha_0|0\rangle - \alpha_1|1\rangle \xrightarrow{\mathrm{I}_2} \cdot \xrightarrow{\mathrm{Z}} |\psi_Q\rangle$

When QA is measured in $|\beta_{11}\rangle$ then B is in $\alpha_0|1\rangle - \alpha_1|0\rangle \xrightarrow{\mathrm{X}} \cdot \xrightarrow{\mathrm{Z}} |\psi_Q\rangle$.

Princeps references on superdense coding . . .

[1] C. H. Bennett, S. J. Wiesner; "Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states"; *Physical Review Letters* 69 (1992) 2881–2884.

[2] K. Mattle, H. Weinfurter, P. G. Kwiat, and A. Zeilinger; "Dense coding in experimental quantum communication"; *Physical Review Letters* 76 (1996) 4656–4659.

. . . and teleportation

[3] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, W. K. Wootters; "Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels"; *Physical Review Letters* 70 (1993) 1895–1899.

Grover quantum search algorithm (1/3) *Phys. Rev. Lett.* 79 (1997) 325.

• Finds an item out of N in an unsorted database, in $O(\sqrt{N})$ complexity instead of $O(N)$ classically.

• An N-dimensional quantum system with orthonormal basis $\{|1\rangle, \cdots, |N\rangle\}$, the states $|n\rangle$, $n = 1, \dots N$, representing the N items stored in the database.

• A set of N real values $\{\omega_1, \cdots, \omega_N\}$ representing the address of each item $|n\rangle$ in the database.

• The unsorted database is in the state $|\psi\rangle = \dfrac{1}{\sqrt{N}} \sum_{n=1}^{N} |n\rangle$.

• A query of the database, in order to obtain the address ω_n of an item $|n\rangle$, is performed by a measurement of the observable $\Omega = \sum_{n=1}^{N} \omega_n |n\rangle\langle n|$.

• Any specific item $|n_0\rangle$ is obtained as measurement outcome with its eigenvalue (address) ω_{n_0}, with the probability $|\langle n_0|\psi\rangle|^2 = 1/N$ (since $\langle n_0|\psi\rangle = 1/\sqrt{N}$).

Grover quantum search algorithm (2/3)

• For this specific item $|n_0\rangle$ that we want to retrieve (obtain its address ω_{n_0}), it is possible to amplify this uniform probability $|\langle n_0|\psi\rangle|^2 = 1/N$.

• Let $|n_\perp\rangle = \dfrac{1}{\sqrt{N-1}} \sum_{n\neq n_0}^{N} |n\rangle$ normalized state $\perp |n_0\rangle \Longrightarrow |\psi\rangle$ in plane $(|n_0\rangle, |n_\perp\rangle)$.

• Define unitary operator $\mathrm{U}_0 = \mathrm{I}_N - 2|n_0\rangle\langle n_0| \Longrightarrow \mathrm{U}_0|n_\perp\rangle = |n_\perp\rangle$ and $\mathrm{U}_0|n_0\rangle = -|n_0\rangle$.
So in plane $(|n_0\rangle, |n_\perp\rangle)$, the operator U_0 performs a reflection about $|n_\perp\rangle$. (U_0 oracle).

• Let $|\psi_\perp\rangle$ normalized state $\perp |\psi\rangle$ in plane $(|n_0\rangle, |n_\perp\rangle)$.

• Define the unitary operator $\mathrm{U}_\psi = 2|\psi\rangle\langle\psi| - \mathrm{I}_N \Longrightarrow \mathrm{U}_\psi|\psi\rangle = |\psi\rangle$ and $\mathrm{U}_\psi|\psi_\perp\rangle = -|\psi_\perp\rangle$.
So in plane $(|n_0\rangle, |n_\perp\rangle)$, the operator U_ψ performs a reflection about $|\psi\rangle$.

• In plane $(|n_0\rangle, |n_\perp\rangle)$, the composition of two reflections is a rotation $\mathrm{U}_\psi\mathrm{U}_0 = \mathrm{G}$ (Grover amplification operator). It verifies $\mathrm{G}|n_0\rangle = \mathrm{U}_\psi\mathrm{U}_0|n_0\rangle = -\mathrm{U}_\psi|n_0\rangle = |n_0\rangle - \dfrac{2}{\sqrt{N}}|\psi\rangle$.
The rotation angle θ between $|n_0\rangle$ and $\mathrm{G}|n_0\rangle$, via the scalar product of $|n_0\rangle$ and $\mathrm{G}|n_0\rangle$, verifies $\cos(\theta) = \langle n_0|\mathrm{G}|n_0\rangle = 1 - \dfrac{2}{N} \approx 1 - \dfrac{\theta^2}{2} \Longrightarrow \theta \approx \dfrac{2}{\sqrt{N}}$ at $N \gg 1$.

Grover quantum search algorithm (3/3)

- In plane $(|n_0\rangle, |n_\perp\rangle)$, the rotation $\mathrm{G} = \mathrm{U}_\psi \mathrm{U}_0$ is with angle $\theta \approx \dfrac{2}{\sqrt{N}}$.

- $\mathrm{G}|\psi\rangle = \mathrm{U}_\psi \mathrm{U}_0 |\psi\rangle = \mathrm{U}_\psi\Big(|\psi\rangle - \dfrac{2}{\sqrt{N}}|n_0\rangle\Big) = \Big(1 - \dfrac{4}{N}\Big)|\psi\rangle + \dfrac{2}{\sqrt{N}}|n_0\rangle$.

So after rotation by θ the rotated state $\mathrm{G}|\psi\rangle$ is closer to $|n_0\rangle$.

- $\mathrm{G}|\psi\rangle$ remains in plane $(|n_0\rangle, |n_\perp\rangle)$, and any state in plane $(|n_0\rangle, |n_\perp\rangle)$ by G is rotated by θ.

So $\mathrm{G}^2|\psi\rangle$ rotates $|\psi\rangle$ by 2θ toward $|n_0\rangle$, and $\mathrm{G}^k|\psi\rangle$ rotates $|\psi\rangle$ by $k\theta$ toward $|n_0\rangle$.

- The angle Θ of $|\psi\rangle$ and $|n_0\rangle$ is such that $\cos(\Theta) = \langle n_0|\psi\rangle = 1/\sqrt{N} \Longrightarrow \Theta = \mathrm{acos}(1/\sqrt{N})$.

- So $K = \dfrac{\Theta}{\theta} \approx \dfrac{\sqrt{N}}{2}\mathrm{acos}(1/\sqrt{N})$ iterations of G rotate $|\psi\rangle$ onto $|n_0\rangle$.

At most $\Theta = \dfrac{\pi}{2} \Longrightarrow$ at most $K \approx \dfrac{\pi}{4}\sqrt{N}$.

- So when the state $\mathrm{G}^K|\psi\rangle \approx |n_0\rangle$ is measured, the probability is almost 1 to obtain $|n_0\rangle$ and its address $\omega_{n_0} \Longrightarrow$ The searched item is found in $O(\sqrt{N})$ operations instead of $O(N)$ classically.

Other quantum algorithms

- Shor factoring algorithm (1997) :

Factors any integer in polynomial complexity
(instead of exponential classically).

$15 = 3 \times 5$, with spin-1/2 nuclei (Vandersypen *et al.*, Nature 2001).

$21 = 3 \times 7$, with photons (Martín-López *et al.*, Nature Photonics 2012).

- http://math.nist.gov/quantum/zoo/

"A comprehensive catalog of quantum algorithms . . . "

Quantum correlations (1/2)

Alice and Bob share a pair of qubits in the entangled (Bell) state $|\psi_{\mathrm{AB}}\rangle = \dfrac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big)$.

Alice or Bob on its qubit can measure observables of the form $\Omega(\theta) = \sin(\theta)\mathrm{X} + \cos(\theta)\mathrm{Z}$, having eigenvalues ± 1.

Alice measures $\Omega(\alpha)$ to obtain $A = \pm 1$, and Bob measures $\Omega(\beta)$ to obtain $B = \pm 1$, then from $\rho_{\mathrm{AB}} = |\psi_{\mathrm{AB}}\rangle\langle\psi_{\mathrm{AB}}|$ we obtain the average $\langle AB\rangle = \mathrm{tr}\Big(\rho_{\mathrm{AB}}\Omega(\alpha)\otimes\Omega(\beta)\Big) = -\cos(\alpha - \beta)$.

For any four random variables A_1, A_2, B_1, B_2 with values ± 1,
$\Gamma = (A_1 + A_2)B_1 - (A_1 - A_2)B_2 = A_1B_1 + A_2B_1 + A_2B_2 - A_1B_2 = \pm 2$,
because since $A_1, A_2 = \pm 1$, either $(A_1 + A_2)B_1 = 0$ or $(A_1 - A_2)B_2 = 0$,
and in each case the remaining term is ± 2.

So for any probability distribution on (A_1, A_2, B_1, B_2), necessarily
$\langle\Gamma\rangle = \langle A_1B_1 + A_2B_1 + A_2B_2 - A_1B_2\rangle = \langle A_1B_1\rangle + \langle A_2B_1\rangle + \langle A_2B_2\rangle - \langle A_1B_2\rangle$
verifies $-2 \leq \langle\Gamma\rangle \leq 2$. Bell inequalities (1964).

Quantum correlations (2/2)

A long series of experiments repeated on identical copies of $|\psi_{\mathrm{AB}}\rangle$:
EPR experiment (Einstein, Podolsky, Rosen, 1935).

Alice chooses to randomly switch between measuring $\mathrm{A}_1 \equiv \Omega(\alpha_1)$ or $\mathrm{A}_2 \equiv \Omega(\alpha_2)$,
and Bob chooses to randomly switch between measuring $\mathrm{B}_1 \equiv \Omega(\beta_1)$ or $\mathrm{B}_2 \equiv \Omega(\beta_2)$.

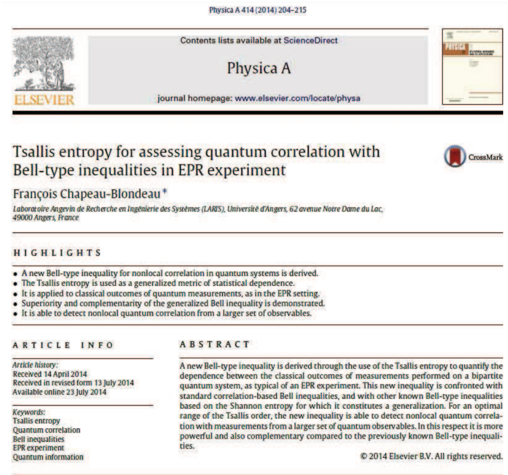
For $\langle\Gamma\rangle = \langle A_1B_1\rangle + \langle A_2B_1\rangle + \langle A_2B_2\rangle - \langle A_1B_2\rangle$ one obtains
$\langle\Gamma\rangle = -\cos(\alpha_1 - \beta_1) - \cos(\alpha_2 - \beta_1) - \cos(\alpha_2 - \beta_2) + \cos(\alpha_1 - \beta_2)$.

The choice $\alpha_1 = 0,\ \alpha_2 = \pi/2$ and $\beta_1 = \pi/4,\ \beta_2 = 3\pi/4$ leads to
$\langle\Gamma\rangle = -\cos(\pi/4) - \cos(\pi/4) - \cos(\pi/4) + \cos(3\pi/4) = -2\sqrt{2} < -2$.

Bell inequalities are violated by quantum measurements.

Experimentally verified (Aspect *et al.*, Phys. Rev. Lett. 1981, 1982).

Local realism and separability (classical) replaced by
a nonlocal nonseparable reality (quantum).



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Tsallis entropy for assessing quantum correlation with Bell-type inequalities in EPR experiment

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HIGHLIGHTS

- A new Bell-type inequality for nonlocal correlation in quantum systems is derived.
- The Tsallis entropy is used as a generalized metric of statistical dependence.
- It is applied to classical outcomes of quantum measurements, as in the EPR setting.
- Superiority and complementarity of the generalized Bell inequality is demonstrated.
- It is able to detect nonlocal quantum correlation from a larger set of observables.

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ABSTRACT

A new Bell-type inequality is derived through the use of the Tsallis entropy to quantify the dependence between the classical outcomes of measurements performed on a bipartite quantum system, as typical of an EPR experiment. This new inequality is confronted with standard correlation-based Bell inequalities, and with other known Bell-type inequalities based on the Shannon entropy for which it constitutes a generalization. For an optimal range of the Tsallis order, the new inequality is able to detect nonlocal quantum correlation with measurements from a larger set of quantum observables. In this respect it is more powerful and also complementary compared to the previously known Bell-type inequalities.

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GHZ states (1/5) (1989, Greenberger, Horne, Zeilinger)

Three players, each receiving a binary input $x_j = 0/1$, for $j = 1, 2, 3$,
with four possible input configurations $x_1x_2x_3 \in \{000, 011, 101, 110\}$.

Each player j responds by a binary output $y_j(x_j) = 0/1$, function only of its own input x_j, for $j = 1, 2, 3$.

Game is won if the players collectively respond according to the input–output matches :

$x_1x_2x_3 = 000 \longrightarrow y_1y_2y_3$ such that $y_1 \oplus y_2 \oplus y_3 = 0$,
$x_1x_2x_3 \in \{011, 101, 110\} \longrightarrow y_1y_2y_3$ such that $y_1 \oplus y_2 \oplus y_3 = 1$.

To select their responses $y_j(x_j)$, the players can agree on a collective strategy before, but not after, they have received their inputs x_j.

GHZ states (2/5)

A strategy winning on all four input configurations
would consist in three binary functions $y_j(x_j)$ meeting the four constraints :

$$\begin{aligned} y_1(0) \oplus y_2(0) \oplus y_3(0) &= 0 \\ y_1(0) \oplus y_2(1) \oplus y_3(1) &= 1 \\ y_1(1) \oplus y_2(0) \oplus y_3(1) &= 1 \\ y_1(1) \oplus y_2(1) \oplus y_3(0) &= 1 \end{aligned}$$

$0 \oplus 0 \oplus 0 = 1$, by summation of the four constraints,
$\Longrightarrow$ $0 = 1$, so the four constraints are incompatible.

So no (classical) strategy exists that would win on all four input configurations.
Any (classical) strategy is bound to fail on some input configuration(s).

We show a strategy using quantum resources winning on all four input configurations,
(by escaping local realism, $y_j(0) = 0/1$ and $y_j(1) = 0/1$ not existing simultaneously).

GHZ states (3/5)

Before the game starts, each player receives one qubit from a qubit triplet prepared in the entangled state (GHZ state)

$$|\psi\rangle = |\psi_{123}\rangle = \frac{1}{2}\Big(|000\rangle - |011\rangle - |101\rangle - |110\rangle\Big).$$

And the players agree on the common (prior) strategy :
if $x_j = 0$, player j obtains y_j as the outcome of measuring its qubit in basis $\{|0\rangle, |1\rangle\}$,
if $x_j = 1$, player j obtains y_j as the outcome of measuring its qubit in basis $\{|+\rangle, |-\rangle\}$.

We prove this is a winning strategy on all four input configurations :

1) When $x_1x_2x_3 = 000$, the three players measure in $\{|0\rangle, |1\rangle\}$
$\Longrightarrow y_1 \oplus y_2 \oplus y_3 = 0$ is matched.

GHZ states (4/5)

2) When $x_1x_2x_3 = 011$, only player 1 measures in $\{|0\rangle, |1\rangle\}$.

$|\psi\rangle = \frac{1}{2}\Big(|000\rangle - |011\rangle - |101\rangle - |110\rangle\Big) = \frac{1}{2}\Big[|0\rangle\Big(|00\rangle - |11\rangle\Big) - |1\rangle\Big(|01\rangle + |10\rangle\Big)\Big]$.

Since $|0\rangle = \frac{1}{\sqrt{2}}\Big(|+\rangle + |-\rangle\Big)$, $|1\rangle = \frac{1}{\sqrt{2}}\Big(|+\rangle - |-\rangle\Big) \Longrightarrow$

$$\begin{aligned} |00\rangle - |11\rangle &= \frac{1}{2}\Big[\Big(|+\rangle + |-\rangle\Big)\Big(|+\rangle + |-\rangle\Big) - \Big(|+\rangle - |-\rangle\Big)\Big(|+\rangle - |-\rangle\Big)\Big] \\ &= \frac{1}{2}\Big[\Big(|{+}{+}\rangle + |{+}{-}\rangle + |{-}{+}\rangle + |{-}{-}\rangle\Big) - \Big(|{+}{+}\rangle - |{+}{-}\rangle - |{-}{+}\rangle + |{-}{-}\rangle\Big)\Big] \\ &= |{+}{-}\rangle + |{-}{+}\rangle\,; \end{aligned}$$

$|01\rangle + |10\rangle = \frac{1}{2}\Big[\Big(|+\rangle + |-\rangle\Big)\Big(|+\rangle - |-\rangle\Big) + \Big(|+\rangle - |-\rangle\Big)\Big(|+\rangle + |-\rangle\Big)\Big] = |{+}{+}\rangle - |{-}{-}\rangle$;

$\Longrightarrow |\psi\rangle = \frac{1}{2}\Big(|0+-\rangle + |0-+\rangle - |1++\rangle + |1--\rangle\Big) \Longrightarrow y_1 \oplus y_2 \oplus y_3 = 1$ matched.

GHZ states (5/5)

3) When $x_1x_2x_3 = 101$, only player 2 measures in $\{|0\rangle, |1\rangle\}$.

$$|\psi\rangle = \frac{1}{2}\Big(|000\rangle - |011\rangle - |101\rangle - |110\rangle\Big) = \frac{1}{2}\Big[|{\cdot}0{\cdot}\rangle\Big(|0\cdot 0\rangle - |1\cdot 1\rangle\Big) - |{\cdot}1{\cdot}\rangle\Big(|0\cdot 1\rangle + |1\cdot 0\rangle\Big)\Big]$$

$$= \frac{1}{2}\Big[|{\cdot}0{\cdot}\rangle\Big(|+\cdot-\rangle + |-\cdot+\rangle\Big) - |{\cdot}1{\cdot}\rangle\Big(|+\cdot+\rangle - |-\cdot-\rangle\Big)\Big]$$

$$= \frac{1}{2}\Big(|+0-\rangle + |-0+\rangle - |+1+\rangle + |-1-\rangle\Big) \Longrightarrow y_1 \oplus y_2 \oplus y_3 = 1 \ \text{ matched.}$$

4) When $x_1x_2x_3 = 110$, only player 3 measures in $\{|0\rangle, |1\rangle\}$.

$$|\psi\rangle = \frac{1}{2}\Big(|000\rangle - |011\rangle - |101\rangle - |110\rangle\Big) = \frac{1}{2}\Big[|{\cdot}{\cdot}0\rangle\Big(|00{\cdot}\rangle - |11{\cdot}\rangle\Big) - |{\cdot}{\cdot}1\rangle\Big(|01{\cdot}\rangle + |10{\cdot}\rangle\Big)\Big]$$

$$= \frac{1}{2}\Big[|{\cdot}{\cdot}0\rangle\Big(|+-{\cdot}\rangle + |-+{\cdot}\rangle\Big) - |{\cdot}{\cdot}1\rangle\Big(|++{\cdot}\rangle - |--{\cdot}\rangle\Big)\Big]$$

$$= \frac{1}{2}\Big(|+-0\rangle + |-+0\rangle - |++1\rangle + |--1\rangle\Big) \Longrightarrow y_1 \oplus y_2 \oplus y_3 = 1 \ \text{ matched.}$$

Density operator (1/2)

Quantum system in (pure) state $|\psi_j\rangle$, measured in an orthonormal basis $\{|n\rangle\}$:

$\Longrightarrow$ probability $\Pr\{|n\rangle \big| |\psi_j\rangle\} = |\langle n|\psi_j\rangle|^2 = \langle n|\psi_j\rangle\langle\psi_j|n\rangle$.

Several possible states $|\psi_j\rangle$ with probabilities p_j (with $\sum_j p_j = 1$) :

$\Longrightarrow \Pr\{|n\rangle\} = \sum_j p_j \Pr\{|n\rangle \big| |\psi_j\rangle\} = \langle n| \Big(\sum_j p_j |\psi_j\rangle\langle\psi_j|\Big) |n\rangle = \langle n|\rho|n\rangle$,

with density operator $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$.

and $\Pr\{|n\rangle\} = \langle n|\rho|n\rangle = \mathrm{tr}(\rho|n\rangle\langle n|) = \mathrm{tr}(\rho\,\Pi_n)$.

The quantum system is in a **mixed** state, corresponding to the statistical ensemble $\{p_j, |\psi_j\rangle\}$, described by the density operator ρ.

Lemma : For any operator A with trace $\mathrm{tr}(\mathrm{A}) = \sum_n \langle n|\mathrm{A}|n\rangle$, one has
$\mathrm{tr}(\mathrm{A}|\psi\rangle\langle\phi|) = \sum_n \langle n|\mathrm{A}|\psi\rangle\langle\phi|n\rangle = \sum_n \langle\phi|n\rangle\langle n|\mathrm{A}|\psi\rangle = \langle\phi|\Big(\sum_n |n\rangle\langle n|\Big)\mathrm{A}|\psi\rangle = \langle\phi|\mathrm{A}|\psi\rangle$

Density operator (2/2)

Density operator $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$

$\Longrightarrow \rho = \rho^\dagger$ Hermitian ;

$\forall |\psi\rangle, \langle\psi|\rho|\psi\rangle = \sum_j p_j |\langle\psi|\psi_j\rangle|^2 \geq 0 \Longrightarrow \rho \geq 0$ positive ;

trace $\mathrm{tr}(\rho) = \sum_j p_j \mathrm{tr}(|\psi_j\rangle\langle\psi_j|) = \sum_j p_j = 1$.

On $\mathcal{H}_N$, eigen decomposition $\rho = \sum_{n=1}^{N} \lambda_n |\lambda_n\rangle\langle\lambda_n|$, with

eigenvalues $\{\lambda_n\}$ a probability distribution,
eigenstates $\{|\lambda_n\rangle\}$ an orthonormal basis of $\mathcal{H}_N$.

Purity $\mathrm{tr}(\rho^2) = \sum_{n=1}^{N} \lambda_n^2 = 1$ for a pure state, and $\mathrm{tr}(\rho^2) < 1$ for a mixed state.

A valid density operator on $\mathcal{H}_N \equiv$ any positive operator ρ with unit trace, provides a general representation for the state of a quantum system in $\mathcal{H}_N$.

Average of an observable

A quantum system in $\mathcal{H}_N$ has observable Ω of diagonal form $\Omega = \sum_{n=1}^{N} \omega_n |\omega_n\rangle\langle\omega_n|$.

When the quantum system is in state ρ, measuring Ω amounts to performing a projective measurement on ρ in the orthonormal eigenbasis $\{|\omega_1\rangle, \dots |\omega_N\rangle\}$ of $\mathcal{H}_N$, with the N orthogonal projectors $|\omega_n\rangle\langle\omega_n|$, for $n = 1$ to N.

The outcome yields the eigenvalue $\omega_n \in \mathbb{R}$ with probability
$\Pr\{\omega_n\} = \langle\omega_n|\rho|\omega_n\rangle = \mathrm{tr}(\rho|\omega_n\rangle\langle\omega_n|)$.

Over repeated measurements of Ω on the system prepared in the same state ρ, the average value of Ω is

$$\langle\Omega\rangle = \sum_{n=1}^{N} \omega_n \Pr\{\omega_n\} = \sum_{n=1}^{N} \omega_n \mathrm{tr}(\rho|\omega_n\rangle\langle\omega_n|) = \mathrm{tr}\Big(\rho \sum_{n=1}^{N} \omega_n |\omega_n\rangle\langle\omega_n|\Big)$$
$$= \mathrm{tr}(\rho\Omega).$$

Density operator for the qubit

$\{\sigma_0 = \mathrm{I}_2, \sigma_x, \sigma_y, \sigma_z\}$ a basis of $\mathcal{H}_2$,
orthogonal for the Hilbert-Schmidt inner product $\mathrm{tr}(\mathrm{A}^\dagger\mathrm{B})$.

Any $\rho = \dfrac{1}{2}(\mathrm{I}_2 + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z) = \dfrac{1}{2}(\mathrm{I}_2 + \vec{r}\,\vec{\sigma})$.

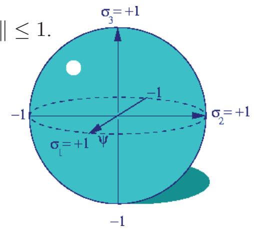
$\Longrightarrow \mathrm{tr}(\rho) = 1$.

$\rho = \rho^\dagger \Longrightarrow r_x = r_x^*, \ \ r_y = r_y^*, \ \ r_z = r_z^* \Longrightarrow r_x, r_y, r_z$ real.

Eigenvalues $\lambda_\pm = \dfrac{1}{2}(1 \pm \|\vec{r}\|) \geq 0 \Longrightarrow \|\vec{r}\| \leq 1$.

$\|\vec{r}\| < 1$ for mixed states,
$\|\vec{r}\| = 1$ for pure states.

$\vec{r} = [r_x, r_y, r_z]^\top$ in Bloch ball of $\mathbb{R}^3$.



Observables on the qubit

Any operator on $\mathcal{H}_2$ has general form $\Omega = a_0\mathrm{I}_2 + \vec{a}\,\vec{\sigma}$,
with determinant $\det(\Omega) = a_0^2 - \vec{a}^{\,2}$, two eigenvalues $a_0 \pm \sqrt{\vec{a}^{\,2}}$,

and two projectors on the two eigenvectors $|\pm\vec{a}\rangle\langle\pm\vec{a}| = \dfrac{1}{2}\Big(\mathrm{I}_2 \pm \vec{a}\,\vec{\sigma}/\sqrt{\vec{a}^{\,2}}\Big)$.

For an observable, Ω Hermitian requires $a_0 \in \mathbb{R}$ and $\vec{a} = [a_x, a_y, a_z]^\top \in \mathbb{R}^3$.

An important observable measurable on the qubit is $\Omega = \vec{a}\,\vec{\sigma}$ with $\|\vec{a}\| = 1$,
known as a spin measurement in the direction $\vec{a}$ of $\mathbb{R}^3$,
yielding as possible outcomes the two eigenvalues $\pm\|\vec{a}\| = \pm 1$,
with probabilites $\Pr\{\pm 1\} = \dfrac{1}{2}\Big(1 \pm \vec{r}\,\vec{a}\Big)$ for a qubit in state $\rho = \dfrac{1}{2}\Big(\mathrm{I}_2 + \vec{r}\,\vec{\sigma}\Big)$,

$\Big($since $\Pr\{\pm 1\} = \mathrm{tr}\Big(\rho|\pm\vec{a}\rangle\langle\pm\vec{a}|\Big) = \dfrac{1}{2} \pm \dfrac{1}{2}\mathrm{tr}(\rho\vec{a}\vec{\sigma})$ with $(\vec{r}\,\vec{\sigma})(\vec{a}\,\vec{\sigma}) = (\vec{r}\,\vec{a})\mathrm{I}_2 + i(\vec{r}\times\vec{a})\vec{\sigma}\Big)$.

Generalized measurement

In a Hilbert space $\mathcal{H}_N$ with dimension N, the state of a quantum system is specified by a Hermitian positive unit-trace density operator ρ.

• Projective measurement :

Defined by a set of N orthogonal projectors $|n\rangle\langle n| = \Pi_n$,

verifying $\sum_n |n\rangle\langle n| = \sum_n \Pi_n = \mathrm{I}_N$,

and $\Pr\{|n\rangle\} = \mathrm{tr}(\rho\Pi_n)$. Moreover $\sum_n \Pr\{|n\rangle\} = 1, \forall\rho \Longleftrightarrow \sum_n \Pi_n = \mathrm{I}_N$.

• Generalized measurement :

Defined by a set of an arbitrary number of positive operators M_m,

verifying $\sum_m \mathrm{M}_m = \mathrm{I}_N$,

and $\Pr\{\mathrm{M}_m\} = \mathrm{tr}(\rho\mathrm{M}_m)$. Moreover $\sum_m \Pr\{\mathrm{M}_m\} = 1, \forall\rho \Longleftrightarrow \sum_m \mathrm{M}_m = \mathrm{I}_N$.

Quantum noise (1/2)

A quantum system of $\mathcal{H}_N$ in state ρ interacting with its environment represents an open quantum system. The state ρ usually undergoes a nonunitary evolution.

With ρ_{env} the state of the environment at the onset of the interaction, the joint state $\rho\otimes\rho_{\mathrm{env}}$ can be considered as that of a closed system, undergoing a unitary evolution by U as $\rho\otimes\rho_{\mathrm{env}} \longrightarrow \mathrm{U}(\rho\otimes\rho_{\mathrm{env}})\mathrm{U}^\dagger$.

At the end of the interaction, the state of the quantum system of interest is obtained by the partial trace over the environment : $\rho \longrightarrow \mathcal{N}(\rho) = \mathrm{tr}_{\mathrm{env}}\Big[\mathrm{U}(\rho\otimes\rho_{\mathrm{env}})\mathrm{U}^\dagger\Big]$. (1)

Very often, the environment incorporates a huge number of degrees of freedom, and is largely uncontrolled ; it can be understood as quantum noise inducing decoherence.

A very nice feature is that, independently of the complexity of the environment, Eq. (1) can always be put in the form $\rho \longrightarrow \mathcal{N}(\rho) = \sum_\ell \Lambda_\ell\rho\Lambda_\ell^\dagger$ operator-sum or Kraus representation, with the Kraus operators Λ_ℓ, which need not be more than N^2, satisfying $\sum_\ell \Lambda_\ell^\dagger\Lambda_\ell = \mathrm{I}_N$.

Quantum noise (2/2)

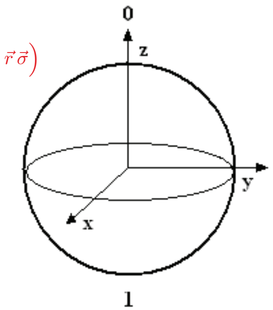
A general transformation of a quantum state ρ can be expressed by the quantum operation $\rho \longrightarrow \mathcal{N}(\rho) = \sum_\ell \Lambda_\ell\rho\Lambda_\ell^\dagger$, with $\sum_\ell \Lambda_\ell^\dagger\Lambda_\ell = \mathrm{I}_N$,
representing a linear completely positive trace-preserving map,
mapping a density operator on $\mathcal{H}_N$ into a density operator on $\mathcal{H}_N$.

For an arbitrary qubit state defined by $\rho = \dfrac{1}{2}\Big(\mathrm{I}_2 + \vec{r}\,\vec{\sigma}\Big)$

with $\|\vec{r}\| \leq 1$,

this is equivalent to the affine map $\vec{r} \rightarrow A\vec{r} + \vec{c}$,

with A a 3×3 real matrix
and $\vec{c}$ a real vector in $\mathbb{R}^3$,
mapping the Bloch ball onto itself.



Quantum noise on the qubit (1/4)

Quantum noise on a qubit in state ρ can be represented by random applications of some of the 4 Pauli operators $\{\mathrm{I}_2, \mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$ on the qubit, e.g.

Bit-flip noise : flips the qubit state with probability p by applying X, or leaves the qubit unchanged with probability $1-p$:

$$\rho \longrightarrow \mathcal{N}(\rho) = (1-p)\rho + p\mathrm{X}\rho\mathrm{X}^\dagger\,, \qquad \vec{r} \longrightarrow A\vec{r} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-2p & 0 \\ 0 & 0 & 1-2p \end{bmatrix} \vec{r}\,.$$

Phase-flip noise : flips the qubit phase with probability p by applying Z, or leaves the qubit unchanged with probability $1-p$:

$$\rho \longrightarrow \mathcal{N}(\rho) = (1-p)\rho + p\mathrm{Z}\rho\mathrm{Z}^\dagger\,, \qquad \vec{r} \longrightarrow A\vec{r} = \begin{bmatrix} 1-2p & 0 & 0 \\ 0 & 1-2p & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{r}\,.$$

Quantum noise on the qubit (2/4)

Depolarizing noise : leaves the qubit unchanged with probability $1-p$, or apply any of X, Y or Z with equal probability $p/3$:

$$\rho \longrightarrow \mathcal{N}(\rho) = (1-p)\rho + \frac{p}{3}\Big(\mathrm{X}\rho\mathrm{X}^\dagger + \mathrm{Y}\rho\mathrm{Y}^\dagger + \mathrm{Z}\rho\mathrm{Z}^\dagger\Big)\,,$$

$$\vec{r} \longrightarrow A\vec{r} = \begin{bmatrix} 1-\frac{4}{3}p & 0 & 0 \\ 0 & 1-\frac{4}{3}p & 0 \\ 0 & 0 & 1-\frac{4}{3}p \end{bmatrix} \vec{r}\,.$$

Quantum noise on the qubit (3/4)

Amplitude damping noise : relaxes the excited state $|1\rangle$ to the ground state $|0\rangle$ with probability γ (for instance by losing a photon) :

$$\rho \longrightarrow \mathcal{N}(\rho) = \Lambda_1\rho\Lambda_1^\dagger + \Lambda_2\rho\Lambda_2^\dagger\,,$$

with $\Lambda_2 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} = \sqrt{\gamma}\,|0\rangle\langle 1|$ taking $|1\rangle$ to $|0\rangle$ with probability γ,

and $\Lambda_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} = |0\rangle\langle 0| + \sqrt{1-\gamma}\,|1\rangle\langle 1|$ which leaves $|0\rangle$ unchanged

and reduces the probability amplitude of resting in state $|1\rangle$.

$$\Longrightarrow\ \vec{r} \longrightarrow A\vec{r} + \vec{c} = \begin{bmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{bmatrix} \vec{r} + \begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix}.$$

Quantum noise on the qubit (4/4)

Generalized amplitude damping noise : interaction of the qubit with a thermal bath at temperature T :

$$\rho \longrightarrow \mathcal{N}(\rho) = \Lambda_1\rho\Lambda_1^\dagger + \Lambda_2\rho\Lambda_2^\dagger + \Lambda_3\rho\Lambda_3^\dagger + \Lambda_4\rho\Lambda_4^\dagger\,,$$

with $\Lambda_1 = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}$, $\Lambda_2 = \sqrt{p}\begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$, $\quad p, \gamma \in [0,1]$,

$$\Lambda_3 = \sqrt{1-p}\begin{bmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{bmatrix}, \quad \Lambda_4 = \sqrt{1-p}\begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix},$$

$$\Longrightarrow\ \vec{r} \longrightarrow A\vec{r} + \vec{c} = \begin{bmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{bmatrix} \vec{r} + \begin{bmatrix} 0 \\ 0 \\ (2p-1)\gamma \end{bmatrix}.$$

Damping $[0,1] \ni \gamma = 1 - e^{-t/T_1} \to 1$ as the interaction time $t \to \infty$ with the bath of the qubit relaxing to equilibrium $\rho_\infty = p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$, with equilibrium probabilities $p = \exp[-E_0/(k_BT)]/Z$ and $1-p = \exp[-E_1/(k_BT)]/Z$ with $Z = \exp[-E_0/(k_BT)] + \exp[-E_1/(k_BT)]$ governed by the Boltzmann distribution between the two energy levels E_0 of $|0\rangle$ and $E_1 > E_0$ of $|1\rangle$.
$T = 0 \Rightarrow p = 1 \Rightarrow \rho_\infty = |0\rangle\langle 0|\,.\quad T \to \infty \Rightarrow p = 1/2 \Rightarrow \rho_\infty \to (|0\rangle\langle 0| + (|1\rangle\langle 1|)/2 = \mathrm{I}_2/2\,.$

Quantum state discrimination

A quantum system can be in one of two alternative states ρ_0 or ρ_1 with prior probabilities P_0 and $P_1 = 1 - P_0$.

Question : What is the best measurement $\{\mathrm{M}_0, \mathrm{M}_1\}$ to decide with a maximal probability of success P_{suc} ?

Answer : One has $P_{\mathrm{suc}} = P_0\,\mathrm{tr}(\rho_0\mathrm{M}_0) + P_1\,\mathrm{tr}(\rho_1\mathrm{M}_1) = P_0 + \mathrm{tr}(\mathrm{T}\mathrm{M}_1)$,
with the test operator $\mathrm{T} = P_1\rho_1 - P_0\rho_0$.

Then P_{suc} is maximized by $\mathrm{M}_1^{\mathrm{opt}} = \sum_{\lambda_n > 0} |\lambda_n\rangle\langle\lambda_n|$,
the projector on the eigensubspace of T with positive eigenvalues λ_n.

The optimal measurement $\{\mathrm{M}_1^{\mathrm{opt}}, \mathrm{M}_0^{\mathrm{opt}} = \mathrm{I}_N - \mathrm{M}_1^{\mathrm{opt}}\}$
achieves the maximum $P_{\mathrm{suc}}^{\max} = \dfrac{1}{2}\Big(1 + \sum_{n=1}^{N} |\lambda_n|\Big)$. (Helstrom 1976)

Discrimination from noisy qubits

Quantum noise on a qubit in state ρ can be represented by random applications of (one of) the 4 Pauli operators $\{\mathrm{I}_2, \mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$ on the qubit, e.g.

Bit-flip noise : $\rho \longrightarrow \mathcal{N}(\rho) = (1-p)\rho + p\mathrm{X}\rho\mathrm{X}^\dagger$,

Depolarizing noise : $\rho \longrightarrow \mathcal{N}(\rho) = (1-p)\rho + \frac{p}{3}\Big(\mathrm{X}\rho\mathrm{X}^\dagger + \mathrm{Y}\rho\mathrm{Y}^\dagger + \mathrm{Z}\rho\mathrm{Z}^\dagger\Big)$.

With a noisy qubit, discrimination from $\mathcal{N}(\rho_0)$ and $\mathcal{N}(\rho_1)$.

$\longrightarrow$ Impact of the probability p of action of the quantum noise,
on the performance $P_{\mathrm{suc}}^{\max}$ of the optimal detector,
in relation to stochastic resonance and enhancement by noise.
(Chapeau-Blondeau, *Physics Letters A* 378 (2014) 2128-2136.)

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Quantum state discrimination and enhancement by noise

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A B S T R A C T

Discrimination between two quantum states is addressed as a quantum detection process where a measurement with two outcomes is performed and a conclusive binary decision results about the state. The performance is assessed by the overall probability of decision error. Based on the theory of quantum detection, the optimal measurement and its performance are exhibited in general conditions. An application is realized on the qubit, for which generic models of quantum noise can be investigated for their impact on state discrimination from a noisy qubit. The quantum noise acts through random application of Pauli operators on the qubit prior to its measurement. For discrimination from a noisy qubit, various situations are exhibited where reinforcement of the action of the quantum noise can be associated with enhanced performance. Such implications of the quantum noise are analyzed and interpreted in relation to stochastic resonance and enhancement by noise in information processing.

Discrimination among $M > 2$ quantum states

A quantum system can be in one of M alternative states ρ_m, for $m = 1$ to M, with prior probabilities P_m with $\sum_{m=1}^{M} P_m = 1$.

Problem : What is the best measurement $\{\mathrm{M}_m\}$ with M outcomes to decide with a maximal probability of success P_{suc} ?

$\Longrightarrow$ Maximize $P_{\mathrm{suc}} = \sum_{m=1}^{M} P_m\,\mathrm{tr}(\rho_m\mathrm{M}_m)$ according to the M operators M_m,
subject to $0 \le \mathrm{M}_m \le \mathrm{I}_N$ and $\sum_{m=1}^{M} \mathrm{M}_m = \mathrm{I}_N$.

For $M > 2$ this problem is only partially solved, in some special cases.
(Barnett *et al.*, Adv. Opt. Photon. 2009).

Error-free discrimination between $M = 2$ states

Two alternative states ρ_0 or ρ_1 of $\mathcal{H}_N$, with priors P_0 and $P_1 = 1 - P_0$,
are not full-rank in $\mathcal{H}_N$, e.g. $\mathrm{supp}(\rho_0) \subset \mathcal{H}_N \Longleftrightarrow [\mathrm{supp}(\rho_0)]^\perp \supset \{\vec{0}\}$.

If $\mathcal{S}_0 = \mathrm{supp}(\rho_0) \cap [\mathrm{supp}(\rho_1)]^\perp \neq \{\vec{0}\}$, error-free discrimination of ρ_0 is possible.
If $\mathcal{S}_1 = \mathrm{supp}(\rho_1) \cap [\mathrm{supp}(\rho_0)]^\perp \neq \{\vec{0}\}$, error-free discrimination of ρ_1 is possible.

Necessity to find a three-outcome measurement $\{\mathrm{M}_0, \mathrm{M}_1, \mathrm{M}_{\mathrm{unc}}\}$:

Find $0 \le \mathrm{M}_0 \le \mathrm{I}_N$ s.t. $\mathrm{M}_0 = \vec{a}_0\Pi_1$ "proportional" to Π_1 projector on $[\mathrm{supp}(\rho_1)]^\perp$,
and $0 \le \mathrm{M}_1 \le \mathrm{I}_N$ s.t. $\mathrm{M}_1 = \vec{a}_1\Pi_0$ "proportional" to Π_0 projector on $[\mathrm{supp}(\rho_0)]^\perp$,
and $\mathrm{M}_0 + \mathrm{M}_1 \le \mathrm{I}_N \Longleftrightarrow \Big[\mathrm{M}_0 + \mathrm{M}_1 + \mathrm{M}_{\mathrm{unc}} = \mathrm{I}_N$ with $0 \le \mathrm{M}_{\mathrm{unc}} \le \mathrm{I}_N\Big]$,
maximizing $P_{\mathrm{suc}} = P_0\,\mathrm{tr}(\mathrm{M}_0\rho_0) + P_1\,\mathrm{tr}(\mathrm{M}_1\rho_1)$ ($\equiv$ min $P_{\mathrm{unc}} = 1 - P_{\mathrm{suc}}$)

This problem is only partially solved, in some special cases,
(Kleinmann *et al.*, J. Math. Phys. 2010).

Error-free discrimination between $M \geq 2$ states

M alternative states ρ_m of $\mathcal{H}_N$, with prior P_m, for $m = 1, \dots M$;
each ρ_m must be with defective rank $< N$.

For all $m = 1$ to M, define $\mathcal{S}_m = \mathrm{supp}(\rho_m) \cap \overbrace{\Big\{ \bigcap_{\ell \neq m} [\mathrm{supp}(\rho_\ell)]^\perp \Big\}}^{\mathcal{K}_m}$.

For each nontrivial $\mathcal{S}_m \neq \{\vec{0}\}$, then ρ_m can go where none other ρ_ℓ can go.
$\Longrightarrow$ Error-free discrimination of ρ_m is possible,

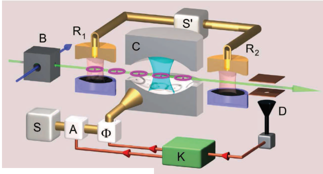
by M_m such that $0 \leq \mathrm{M}_m \leq \mathrm{I}_N$ and M_m "proportional" to the projector on $\mathcal{K}_m$.

To parametrize M_m, find an orthonormal basis $\{|u_j^m\rangle\}_{j=1}^{\dim(\mathcal{K}_m)}$ of $\mathcal{K}_m$,
then $\mathrm{M}_m = \sum_{j=1}^{\dim(\mathcal{K}_m)} a_j^m |u_j^m\rangle \langle u_j^m| = \vec{a}^{\,m}\, \Pi_m$, with Π_m projector on $\mathcal{K}_m$.

Find the M_m (the $\vec{a}^{\,m}$) with $\sum_m \mathrm{M}_m \leq \mathrm{I}_N$ maximizing $P_{\mathrm{suc}} = \sum_m P_m \,\mathrm{tr}(\mathrm{M}_m \rho_m)$.

This problem is only partially solved, in some special cases, (Kleinmann, *J. Math. Phys.* 2010).

Quantum feedback control



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Quantum feedback by discrete quantum nondemolition measurements: Towards on-demand generation of photon-number states

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We propose a quantum feedback scheme for the preparation and protection of photon-number states of light trapped in a high-*Q* microwave cavity. A quantum nondemolition measurement of the cavity field provides information on the photon-number distribution. The feedback loop is closed by injecting into the cavity a coherent pulse adjusted to increase the probability of the target photon number. The efficiency and reliability of the closed-loop state stabilization is assessed by quantum Monte Carlo simulations. We show that, in realistic experimental conditions, the Fock states are efficiently produced and protected against decoherence.

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System dynamics :

• Schrödinger equation (for closed systems)

$$\frac{d}{dt}|\psi\rangle = -\frac{i}{\hbar}\mathrm{H}\,|\psi\rangle \Longrightarrow |\psi(t_2)\rangle = \underbrace{\exp\Big(-\frac{i}{\hbar}\int_{t_1}^{t_2} \mathrm{H}dt\Big)}_{\text{unitary } \mathrm{U}(t_1,t_2)} |\psi(t_1)\rangle = \mathrm{U}(t_1,t_2)\,|\psi(t_1)\rangle$$

Hermitian operator Hamiltonian $\mathrm{H} = \mathrm{H}_0 + \mathrm{H}_u$ (control part H_u).

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[\mathrm{H},\rho] \Longrightarrow \rho(t_2) = \mathrm{U}(t_1,t_2)\,\rho(t_1)\,\mathrm{U}^\dagger(t_1,t_2).$$

• Lindblad equation (for open systems)

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[\mathrm{H},\rho] + \sum_j \Big(2\mathrm{L}_j\rho\mathrm{L}_j^\dagger - \{\mathrm{L}_j^\dagger\mathrm{L}_j, \rho\}\Big),$$ Lindblad op. L_j for interact. with environt.

Measurement : Arbitrary operators $\{\mathrm{E}_m\}$ such that $\sum_m \mathrm{E}_m^\dagger \mathrm{E}_m = \mathrm{I}_N$,
$\Pr\{m\} = \mathrm{tr}(\mathrm{E}_m\rho\mathrm{E}_m^\dagger) = \mathrm{tr}(\rho\mathrm{E}_m^\dagger\mathrm{E}_m) = \mathrm{tr}(\rho\mathrm{M}_m)$ with $\mathrm{M}_m = \mathrm{E}_m^\dagger\mathrm{E}_m$ positive,

Post-measurement state $\rho_m = \dfrac{\mathrm{E}_m\rho\mathrm{E}_m^\dagger}{\mathrm{tr}(\mathrm{E}_m\rho\mathrm{E}_m^\dagger)}$.

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Optimized probing states for qubit phase estimation with general quantum noise

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We exploit the theory of quantum estimation to investigate quantum state estimation in the presence of noise. The quantum Fisher information is used to assess the estimation performance. For the qubit in Bloch representation, general expressions are derived for the quantum score and then for the quantum Fisher information. From this latter expression, it is proved that the Fisher information always increases with the purity of the measured qubit state. An arbitrary quantum noise affecting the qubit is taken into account for its impact on the Fisher information. The task is then specified to estimating the phase of a qubit in a rotation around an arbitrary axis, equivalent to estimating the phase of an arbitrary single-qubit quantum gate. The analysis enables determination of the optimal probing states best resistant to the noise, and proves that they always are pure states but need to be specifically matched to the noise. This optimization is worked out for several noise models important to the qubit. An adaptive scheme and a Bayesian approach are presented to handle phase-dependent solutions.

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Merci de votre attention.

Si vous avez compris . . .
c'est que je me suis mal exprimé !

"Nobody really understands quantum mechanics."
R. P. Feynman