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A MODEL OF RANDOM SIGNAL WITH LONG-RANGE CORRELATIONS

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Abstract : We propose a model which generates a random signal with long-range correlations characterized by a power-law decay. The model takes the form of a first-order recurrence. This form is especially suited for on-line synthesis of long-range correlations that will exist over potentially unlimited ranges. Both theoretical arguments and numerical estimations are given to establish the power-law form of the correlations. The definition of the model incorporates a nonstationary gain, and we focus on how the specification of this gain provides control over the exponent of the power-law decay of the correlations.

1 Introduction

Long-range correlations in random signals are identified by a slow decay, slower than exponential. Typically, these correlations decay according to a power law, confering statistical self-similarity and a fractal character to the signals [1, 2]. Such types of random signals are experimentally observed in a large variety of physical processes, some of them with currently very acute technological interest, and among which are data traffic in computer networks, motorway traffic, fluctuations of financial indices, turbulence, noise in semiconductor devices, neuron activities [2, 3]. The theoretical modeling of such signals with long-range corrrelations remains an important issue, not fully resolved. Relatively few models exist to theoretically construct random signals with long-range correlations and to synthesize actual realizations of them. Fractional Brownian motions [4, 5], or white noises submitted to fractional integration operators [6], offer models for random signal with long-range correlations. Yet, such models are in principle of infinite order, and for practical synthesis of actual realizations, they usually have to be truncated. This generally yields recursive algorithms of high, but finite, order, but synthesizing long-range corrrelations that exist only over a limited range. Other methods, like Cholesky decomposition or wavelet synthesis [7, 8], perform block synthesis instead of recurrent synthesis. When a realization of N points is synthesized, the subsequent addition of one more point with long-range correlations usually requires a new synthesis of a complete block of $N+1$ points from scratch.

Here we present a model which defines a signal with long-range correlations and which takes the form of a simple first-order recurrence. Under this form, the model is especially convenient for on-line synthesis of long-range correlations that will exist over potentially unlimited ranges. In its present form, the model is an extension to a simpler version introduced in [9, 10]. This previous version was able to produce correlations with a power-law decay of the form $\tau^{-\beta}$ in the lag τ, with the exponent β limited to the value 1/2. The present extension of the model allows to continuously span values of β between 0 and 1/2.

2 The model

We consider the dynamical system defined by the first-order recurrence

$$X(k) = X(k-1) + g(k)x(k) \;, \qquad (1)$$

$$Y(k) = \max[Y(k-1),\, X(k)] \;, \qquad (2)$$

$$y(k) = Y(k) - Y(k-1) \;, \qquad (3)$$

for any integer $k > 0$, with initial condition $X(0) = Y(0) = 0$. The quantities $x(k)$ form the input sequence and are independent and identically distributed random variables with zero mean. The sequence $y(k)$ is the output signal that we show to exhibit long-range correlations with a power-law decay measured by an exponent β. The quantity $g(k)$ plays the role of a nonstationary gain applied to the input $x(k)$. In the earlier versions of the model [9, 10] we had $g(k) = 1$ for all k, restricting the power-law correlations to $\beta = 1/2$. Here, we shall introduce and analyze forms of the nonstationary gain $g(k)$ that preserve the long-range correlations in $y(k)$ and allow to reach power-law decay with an exponent β adjustable between 0 and 1/2. We shall specially examine the control over the exponent β via $g(k)$.

The random signal $y(k)$ defined by Eqs. (1)–(3) represents the successive increments of the running maximum of a random walk having increments $g(k)x(k)$.

In the case where $g(k) = 1$ for all k, these input increments are stationary as they reduce to $x(k)$. It is then possible to theoretically prove [10] that the autocorrelation function of $y(k)$ verifies

$$R(k, \tau) = \mathrm{E}[y(k)y(k+\tau)] \propto k^{-1/2}\tau^{-1/2} , \quad k, \tau \text{ large.} \tag{4}$$

This power-law decay of $R(k, \tau)$ identifies long-range correlations in $y(k)$, especially the power law $\tau^{-\beta}$ in the integer lag τ, with exponent $\beta = 1/2$. Similar to the asymptotic properties of standard random walks [11], the long-term structure of $R(k, \tau)$ (at k, τ large) is unaffected by the type of the statistical distribution of the input, or elementary increments, $x(k)$. Binary, Gaussian, Laplacian, uniform distributions for $x(k)$ have been tested in [9, 10] and shown to preserve the long-range correlations as in Eq. (4), and especially the exponent $\beta = 1/2$.

Yet, for a richer model of random signal, it is desirable to depart from the exponent $\beta = 1/2$ while preserving the long-range correlations. We show next that this is possible through the use of a nonstationary gain $g(k)$ in Eqs. (1)–(3).

3 Nonstationary gain

A class of nonstationary gains that we found suitable comes under a power-law form as $g(k) = (k - k_0)^b$. The exponent b of the gain is the parameter that will provide control over the exponent β of the correlations. The origin k_0 has to be adequately handled: k_0 has to be set to k whenever $y(k) > 0$, in order to provide a gain $g(k)$ that regularly grows in a (self-)similar way over each episode where $y(k)$ (the running maximum of the walk $Y(k)$) remains frozen to zero. The explicit implementation of the gain function is realized as exposed in Fig. 1.

$X(0) = Y(0) = 0$
$k = 1;\ k_0 = 0$
<u>Repeat</u>
  $g(k) = (k - k_0)^b$
  $X(k) = X(k-1) + g(k)x(k)$
  $Y(k) = \max[Y(k-1),\ X(k)]$
  $y(k) = Y(k) - Y(k-1)$
  <u>If</u> $y(k) > 0$ <u>then</u> $k_0 \leftarrow k$ <u>End If</u>
  $k \leftarrow k + 1$
<u>Until</u> an exit condition is true

Figure 1: Implementation of the model with nonstationary gain $g(k)$.

A typical evolution of the signal $y(k)$ produced according to Fig. 1, with $b = 0.2$, is represented in Fig. 2, over intervals of increasing duration. Figure 2 reveals the self-similar structure of $y(k)$, with bursts of activity with $y(k) > 0$ separated by intervals of inactivity with $y(k) = 0$, this occuring in a similar way at all scales. In Fig. 2 we took the input $x(k) = \pm 1$ equiprobably, and we stick to this choice everywhere in this paper, since, as we said, the long-range correlations are unaffected by the detail of the distribution of $x(k)$.

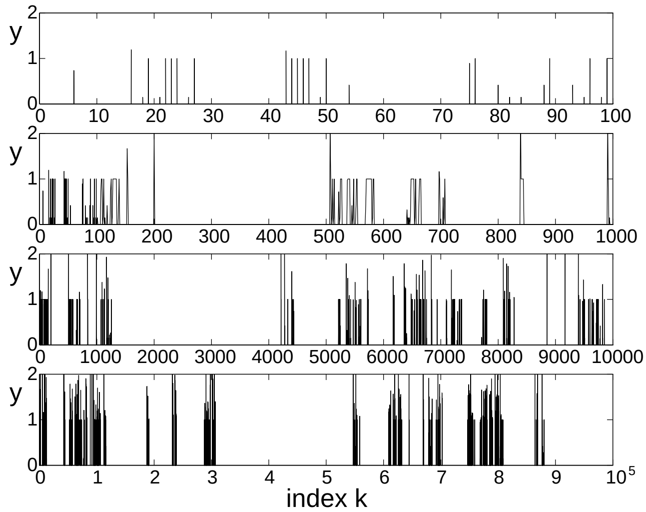

Figure 2: A typical evolution of $y(k)$ produced according to Fig. 1, with $b = 0.2$ and $x(k) = \pm 1$ equiprobably, over intervals of increasing duration revealing the self-similar structure of $y(k)$.

In contrast to the stationary case where $g(k) \equiv 1$, with a nonstationay gain $g(k) = (k - k_0)^b$ it is much more difficult to establish an analytical expression for the autocorrelation function $R(k, \tau) = \mathrm{E}[y(k)y(k+\tau)]$ in order to characterize the long-range dependence. Yet, the recurrent form of Eqs. (1)–(3) allows on-line generation of the signal $y(k)$ over potentially unlimited range, making it easy to perform numerical estimation of the autocorrelation function.

Figure 3 shows, for the nonstationary case, the autocorrelation function of $y(k)$ numerically estimated by the empirical average $N^{-1}\sum_{k=1}^{N} y(k)y(k+\tau)$ over one realization and with $N = 10^7$, as it was done in [10] for the stationary case. Different values of the exponent b of the gain were tested and it was observed, as exemplified by Fig. 3, that the power-law evolution of the autocorrelation function is preserved for any b between 0 and 1/2.

From numerical estimations of the autocorrelation function as in Fig. 3, we measured the slope $-\beta$ of the regression line, for different values of the exponent b. This yielded the evolution of the exponent β of the power-law correlations, as a function of the control parameter provided by the exponent b of the gain $g(k)$. This evolution is represented in Fig. 4. The results of Fig. 4 evidence that the exponent β of the power-law

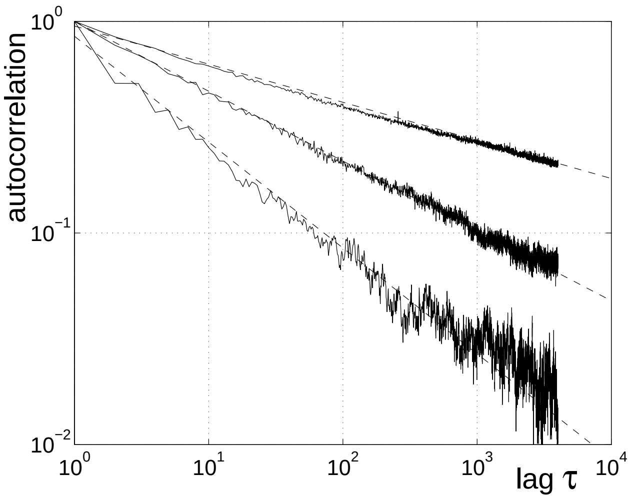

Figure 3: Normalized autocorrelation function of $y(k)$ from Eqs. (1)–(3) numerically estimated for different values of the exponent b of the gain function $g(k)$ of Fig. 1 (solid lines): $b = 0$ (bottom), $b = 0.2$ (middle), $b = 0.4$ (top), together with the regression lines (dashed) of slope $-\beta = -0.50$ (bottom), $-\beta = -0.33$ (middle) and $-\beta = -0.19$ (top), revealing the power-law decay of the correlations.

correlations can be adjusted over a continuous range between 0 and 1/2 by means of the parameter b.

As the parameter b moves above 0.5, we have observed on the numerical estimations, that the autocorrelation function of $y(k)$ tends to gradually depart from a straight line in a log-log plot similar to that of Fig. 3 and tends to give way to a convex curve ($\cup$) instead of a straight line. This trend is illustrated in Fig. 5. Such a behavior still identifies long-range correlations in the signal $y(k)$, but of a more complicated nature this time, characterized by an autocorrelation function that tends to decay more slowly than a power law.

It is to note that the power-law form of the gain function $g(k) = (k - k_0)^b$, implemented as described in Fig. 1, is a key element for obtaining power-law evolutions of the autocorrelation function with a variable exponent β, as depicted in Fig. 3. Especially, the resetting of k_0 whenever $y(k) > 0$, as expressed in Fig. 1, is an essential ingredient. Failing to implement this resetting, by just using instead a power-law gain under the form $g(k) = k^b$ for all k, results in a power-law decay of the autocorrelation that invariably occurs with an exponent $\beta = 0.5$, independent of b, as illustrated by Fig. 6. Also, as visible in Fig. 6, for the gain $g(k) = k^b$ without resetting, the input $g(k)x(k)$, and therefore $y(k)$, can assume large values at large k's, whence the higher variance in the estimation of the autocorrelation of $y(k)$, compared to the case of $g(k) = (k - k_0)^b$ with resetting which limits the excursion of $y(k)$.

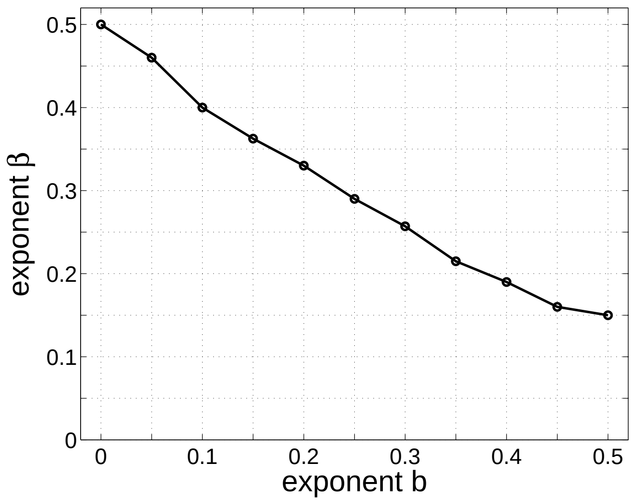

Figure 4: Evolution of the exponent β of the power-law correlations, as a function of the control parameter provided by the exponent b of the nonstationary gain $g(k)$ of Fig. 1.

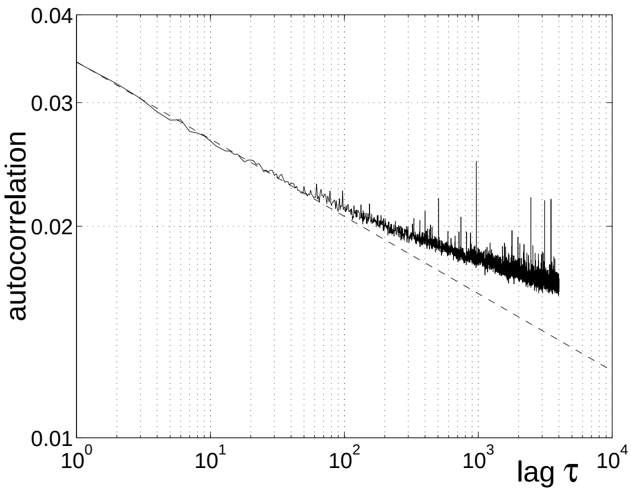

Figure 5: Autocorrelation function of $y(k)$ from Eqs. (1)–(3) numerically estimated with the exponent $b = 0.6$ for the gain function $g(k)$ of Fig. 1 (solid), together with the line of slope -0.11 (dashed).

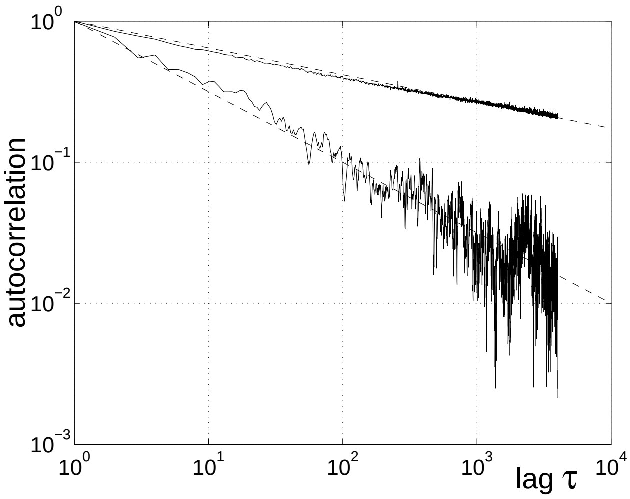

Figure 6: Normalized autocorrelation function of $y(k)$ from Eqs. (1)–(3) numerically estimated for different forms of the gain function $g(k)$ (solid lines): $g(k) = (k - k_0)^{0.4}$ as specified by Fig. 1 (top), $g(k) = k^{0.4}$ for all k (bottom), together with the regression lines (dashed) of slope $-\beta = -0.19$ (top) and $-\beta = -0.5$ (bottom).

4 Conclusion

We have introduced a nonstationary model that was shown capable of generating a random signal $y(k)$ with power-law correlations. A very appealling feature is that this model takes the form of a simple first-order recurrence, with a straightforward numerical implementation, allowing on-line synthesis of long-range correlations over potentially unlimited ranges. Through a numerical study, we have established the possibilty of controlling the exponent of the power-law correlations, by means of a single parameter of the nonstationary model. A theoretical validation aiming at obtaining an explicit analytical expression for the autocorrelation function of $y(k)$ is currently being sought, especially to derive a theoretical form for the "experimental" control law realized by the curve of Fig. 4. Other interesting potentialities lie in the possibility of using $y(k)$ to trigger or modulate secondary stochastic processes, in order to obtain increased flexibility in the long-range dependent signals so produced.

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