Hot Topics in Physical Information (HoTPI-2013)
International Journal of Modern Physics: Conference Series
Vol. 33 (2014) 1460356 (7 pages)

DOI: 10.1142/S2010194514603561

Noise-enhanced transmission efficacy of aperiodic signals in nonlinear systems

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Published 17 September 2014

We study the aperiodic signal transmission in a static nonlinearity in the context of aperiodic stochastic resonance. The performance of a nonlinearity over that of the linear system is defined as the transmission efficacy. The theoretical and numerical results demonstrate that the noise-enhanced transmission efficacy effects occur for different signal strengths in various noise scenarios.

Keywords: Aperiodic signal transmission; noise-enhanced effect; static nonlinearity.

PACS numbers: 05.40.−a, 02.50.−r

1. Introduction

Aperiodic stochastic resonance proposed by Collins *et al.* [1,2] describes the noise-enhanced effect in neural excitable systems for transmitting aperiodic (arbitrary) signals. This notion emphasizes the shape matching between the input and the output signals in biological information processing, and also breaks the limitation of stochastic resonance systems with periodic inputs. In order to characterize the information-carrying signal through the nonlinear system, such measures as the cross-correlation coefficient [1−3], the mutual information [4], the Kullback entropy [5] and the channel capacity [6] are utilized. Later, the notion of aperiodic stochastic resonance, as an outstanding extension of conventional stochastic resonance, inspired its investigation in the area of biological information processing [7–14]. However, some interesting questions have not yet been touched upon. For instance, which type of background noise is favorable for the signal transmission in a nonlinearity? For a linear system, the system performance is easily analyzed. With the performance of a linear system as a benchmark, can we find another system that has a more efficient performance?

In this letter, we mainly focus on aperiodic signal transmission in a static nonlinearity, and also elucidate the above questions. In order to evaluate the performance improvement by noise, the transmission efficacy is defined as the ratio of the cross-correlation coefficient of the output signal from a nonlinear system over that of a linear system. Using this measure, we demonstrate theoretically and numerically that noise-enhanced transmission efficacy effects occur for different signal strengths in various noise scenarios. Especially, in the small-signal limit, the noise type determines the structure of nonlinearity that is locally optimal for an arbitrary signal transmission, and the type of background noise with a high Fisher information is favorable for the signal transmission. Using the illustrative case of a nonlinearity with saturation, efficient signal transmission is demonstrated.

2. Transmission efficacy of nonlinearity

Consider the observation of a process $x(t) = s(t) + z(t)$, where the component $s(t)$ is an information-carrying signal, and zero-mean additive white noise $z(t)$, independent of $s(t)$, having a probability density function (PDF) f_z and variance $\sigma_z^2 = \mathrm{E}_z[z^2] = \int_{-\infty}^{\infty} z^2 f_z(z)dz$. Next, the input $x(t)$ is transmitted by a static nonlinearity

$$y(t) = g(x(t)), \tag{1}$$

where the nonlinearity g has zero mean under f_z, i.e. $\mathrm{E}_z[g(x)] = 0$.

Assume that the known signal $s(t)$ is with a finite non-zero bound U such that $0 \leq |s(t)| \leq U$, and exists for a time duration T. The time average of $s(t)$ is $\langle s(t)\rangle = \int_0^T s(t)dt/T$, and the average power $\langle s^2(t)\rangle = \int_0^T s^2(t)dt/T$ is assumed to be finite. Then, the average signal variance is assumed to be $\sigma_s^2 = \langle s^2(t)\rangle - \langle s(t)\rangle^2$, and the root-mean-square (RMS) amplitude is σ_s. The normalized time average cross correlation between $s(t)$ and $x(t)$ is [1,2]

$$C_{\mathrm{sx}} = \frac{\langle s(t)\mathrm{E}_z[x(t)]\rangle - \langle s(t)\rangle\langle \mathrm{E}_z[x(t)]\rangle}{\sigma_s^2\sqrt{\langle \mathrm{E}_z[x^2(t)]\rangle - \langle \mathrm{E}_z[x(t)]\rangle^2}} = \frac{\sigma_s}{\sqrt{\sigma_s^2+\sigma_z^2}}, \tag{2}$$

where $\langle \mathrm{E}_z[x(t)]\rangle = \langle s(t)\rangle$ and $\langle \mathrm{E}_z[x^2(t)]\rangle = \langle s^2(t)\rangle + \sigma_z^2$. Similarly, the normalized time average cross correlation between $s(t)$ and $y(t)$ is given by [1,2]

$$C_{\mathrm{sy}} = \frac{\langle s(t)\mathrm{E}_z[y(t)]\rangle - \langle s(t)\rangle\langle \mathrm{E}_z[y(t)]\rangle}{\sigma_s\sqrt{\langle \mathrm{E}_z[y^2(t)]\rangle - \langle \mathrm{E}_z[y(t)]\rangle^2}}, \tag{3}$$

with the nonstationary expectations at a fixed time t given by

$$\mathrm{E}_z[y(t)] = \mathrm{E}_z[g(s(t)+z)], \tag{4}$$

$$\mathrm{E}_z[y^2(t)] = \mathrm{E}_z[g^2(s(t)+z)]. \tag{5}$$

For the linear system $g_{\mathrm{L}}(x) = x$, the cross correlation of Eq. (3) is $C_{\mathrm{sy}}^{\mathrm{L}} = C_{\mathrm{sx}}$. In order to evaluate the transmission efficacy of nonlinearity g, the ratio

$$\rho = \frac{C_{\mathrm{sy}}}{C_{\mathrm{sx}}} = \frac{C_{\mathrm{sy}}}{C_{\mathrm{sy}}^{\mathrm{L}}} \tag{6}$$

is an appropriate measure, which represents the improvement of an arbitrary nonlinearity g over the linear system g_{L}.

For instance, consider a typical nonlinearity with saturation [3]

$$g(x) = \tanh(\beta x), \tag{7}$$

with the slope parameter β. An example of the nonlinearity $g(x) = \tanh(5x)$ is shown in Fig. 1. In experiments, $s(t)$ is formed by prefiltering a Gaussian random signal with correlation time τ_s and average signal variance σ_s^2. The autocorrelation of $s(t)$ is $\langle s(t)s(t')\rangle = \sigma_s^2 \exp(-|t-t'|/\tau_s)$. The brackets $\langle\cdot\rangle$ denote an ensemble average [1]. An aperiodic weak signal $s(t)$, as shown in Fig. 2, is adopted with the correlation time $\tau_s = 20$ s, the average signal RMS amplitude $\sigma_s = 0.045$ and total time length $T = 300$ s.

In Fig. 3, we plot the transmission efficacy ρ of saturation nonlinearity $g(x) = \tanh(5x)$ versus Gaussian noise RMS amplitude σ_z for the average signal RMS amplitude $\sigma_s = 0.045, 0.141, 0.447$ and 1.414. It is seen in Fig. 3 that, except for the signal variances $\sigma_s = 0.045$, there is an optimal noise RMS amplitude σ_z that is favorable for signal transmission, this is the aperiodic stochastic resonance effect. This occurs when the signal $s(t)$ is strongly distorted by the saturation nonlinearity. When the noise level is sufficiently large, the signal can be enhanced by noise

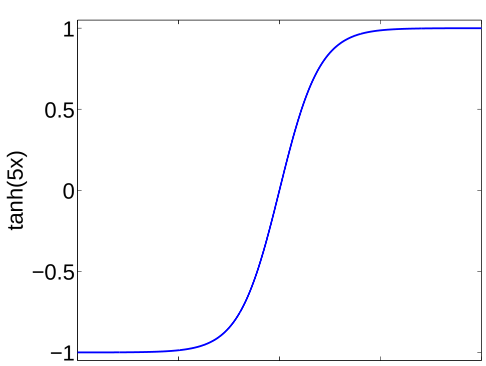

Fig. 1. The nonlinearity $g(x) = \tanh(\beta x)$ with saturation, for the slope parameter $\beta = 5$.

and the distortion of signal is reduced. However, the transmission efficacy ρ of the nonlinearity with saturation is always less than unity in the Gaussian noise background. Can we find the conditions of $\rho > 1$? This problem will be answered in the following.

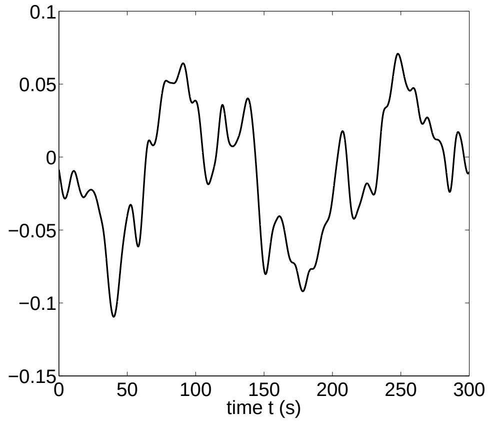

Fig. 2. Aperiodic weak signal $s(t)$ with correlation time $\tau_s = 20$ s, the average signal RMS amplitude $\sigma_s = 0.045$ and total time length 300 s.

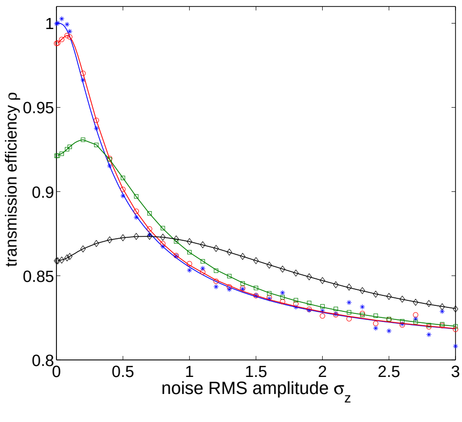

Fig. 3. The transmission efficacy ρ of the nonlinearity with saturation $g(x) = \tanh(5x)$ versus Gaussian noise RMS amplitude σ_z for the average signal RMS amplitude $\sigma_s = 0.045$ (blue), 0.141 (red), 0.447 (green), 1.414 (black).

3. Small-signal limit

It is noted that Eq. (6) is closely dependent on the input signal $s(t)$, and the study of the nonlinearity g becomes an indeterminate case. Consider the weak signal $s(t)$ with the upper bound $U \to 0$ and at a fixed time t, Eq. (4) can be expanded as

$$\mathrm{E}[y(t)] \approx \mathrm{E}_z[g(z) + s(t)g'(z)] = s(t)\mathrm{E}_z[g'(z)], \tag{8}$$

and Eq. (5) can be approximated as

$$\mathrm{E}_z[y_n^2(t)] \approx \mathrm{E}_z\{[g(z) + s(t)g'(z)]^2\} \approx \mathrm{E}_z[g^2(z)] + 2s(t)\mathrm{E}[g(z)g'(z)], \tag{9}$$

up to the first order in the small signal $s(t)$. Substituting Eqs. (8) and (9) int Eq. (3), we have

$$C_{\rm sy} = \frac{\sigma_s \ \mathrm{E}_z[g'(z)]}{\sqrt{\mathrm{E}_z[g^2(z)] + 2s(t)\mathrm{E}[g(z)g'(z)]}} \approx \frac{\sigma_s \ \mathrm{E}_z[g'(z)]}{\sqrt{\mathrm{E}_z[g^2(z)]}}. \tag{10}$$

Therefore, for the linear system $g_{\rm L}(x) = x$, the cross correlation is $C_{\rm sy}^{\rm L} = \sigma_s/\sigma_z$. With the above assumption, the ratio ρ of Eq. (6) becomes

$$\rho = \frac{\sigma_z \ \mathrm{E}_z[g'(z)]}{\sqrt{\mathrm{E}_z[g^2(z)]}} \leq \sigma_z \sqrt{\mathrm{E}\left[\frac{f_z'^2(z)}{f_z^2(z)}\right]} = \sigma_z \sqrt{I(f_z)} = \sqrt{I(f_{z_0})}, \tag{11}$$

where the equality of Eq. (11) occurs as g becomes

$$g_{\rm opt}(z) \triangleq C f_z'(z)/f_z(z), \tag{12}$$

by the Schwarz inequality for the derivative $f_z'(z) = df_z(z)/dz$ (without loss of generality $C = -1$) [15]. Here, the scaled noise $z(t) = \sigma_z z_0(t)$ has PDF $f_z(z) = f_{z_0}(z/\sigma_z)/\sigma_z$, and the standardized noise PDF f_{z_0} is with zero mean and unity variance $\sigma_{z_0}^2 = 1$ [15]. Then, the Fisher information $I(f_z)$ of f_z can be expressed as

$$I(f_z) = \mathrm{E}\left[\frac{f_z'^2(z)}{f_z^2(z)}\right] = \sigma_z^{-2}\mathrm{E}\left[\frac{f_{z_0}'^2(z_0)}{f_{z_0}^2(z_0)}\right] = \sigma_z^{-2} I(f_{z_0}), \tag{13}$$

with the Fisher information $I(f_{z_0})$ of f_{z_0}. This result of Eq. (11) indicates that the Fisher information $I(f_{z_0})$ of noise distribution is closely related to the upper bound of the transmission efficacy ρ.

Since the structure of the locally optimal system $g_{\rm opt}$ in Eq. (12) is established by the knowledge of the noise distribution, and in practice it may be difficult to obtain an explicit analytical expression of $g_{\rm opt}$ in the unknown noisy environment. Thus, we exploit easily implemented nonlinearities to transmit the aperiodic signals. Here, we still consider the saturation nonlinearity of Eq. (1) in the generalized Gaussian noise $z(t)$ with PDF

$$f_z(x) = \frac{c_1}{\sigma_z} \exp\left(-c_2 \left|\frac{x}{\sigma_z}\right|^{\alpha}\right), \tag{14}$$

where $c_1 = \frac{\alpha}{2}\Gamma^{\frac{1}{2}}\left(\frac{3}{\alpha}\right)/\Gamma^{\frac{3}{2}}\left(\frac{1}{\alpha}\right)$ and $c_2 = \left[\Gamma\left(\frac{3}{\alpha}\right)/\Gamma\left(\frac{1}{\alpha}\right)\right]^{\frac{\alpha}{2}}$. For this nonlinearity of Eq. (7), Eq. (11) can be computed as

$$\rho = \frac{\sigma_z \beta \, \mathrm{E}_z[\mathrm{sech}^2(\beta z)]}{\sqrt{1 - \mathrm{E}_z[\mathrm{sech}^2(\beta z)]}}. \tag{15}$$

For different exponents $\alpha = 1$ (Laplacian noise), $1.2, 1.5, 1.7, 2$ (Gaussian noise) and ∞ (uniform noise), the transmission efficiencies ρ of the nonlinearity $g(x) = \tanh(5x)$ are plotted as the function of noise RMS amplitude σ_z by solid lines, as shown in Fig. 4. Here, the average signal RMS amplitude is fixed as $\sigma_s = 0.045$. Each point of the transmission efficacy ρ, as shown in Fig. 4 by datapoints, was averaged over 200 trials in the simulations and the corresponding Gaussian generalized noise is generated with different seeds. It is shown in Fig. 4 theoretically and numerically that the transmission efficacy ρ exhibits the stochastic resonance effect for the exponents $\alpha = 1, 1.2, 1.5, 1.7$. It is noted that, for $\alpha < 2$, the possibility of the transmission efficiencies $\rho > 1$ is demonstrated. However, the maximum of ρ of the nonlinearity with saturation does not yet reach the upper bound of $\sqrt{I(f_{z_0})} = \sqrt{\alpha^2\Gamma\big(3\alpha^{-1}\big)\Gamma\big(2-\alpha^{-1}\big)/\Gamma^2\big(\alpha^{-1}\big)}$ indicated in Eq. (11).

4. Conclusion

In this paper, we studied aperiodic signal transmission in a nonlinearity. The transmission efficacy is adopted as the cross-correlation coefficient of an output signal for a nonlinear system over that of a linear system. Then, we demonstrate theoret-

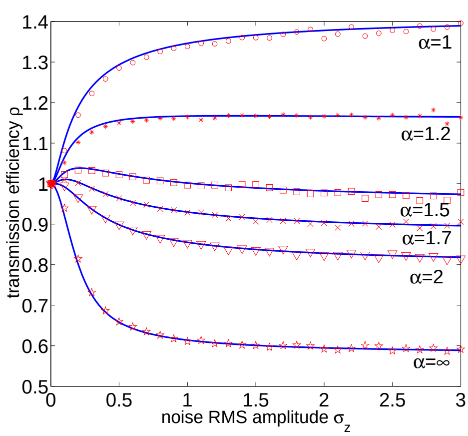

Fig. 4. The transmission efficacy ρ for the nonlinearity $g(x) = \tanh(5x)$ versus the generalized Gaussian noise RMS amplitude σ_z for different exponents $\alpha = 1, 1.2, 1.5, 1.7, 2$ and ∞ (from top to bottom). The solid lines are theoretical curves of Eq. (15), and the data points are the corresponding numerical results. Each point is averaged over 200 trials in the simulations. Here, the average signal RMS amplitude is fixed as $\sigma_s = 0.045$.

ically and numerically that the noise-enhanced transmission efficacy effects occur for different average signal variance, which is illustratively presented by a nonlinearity with saturation. It is found that, in the small-signal limit, the structure of the nonlinearity that is locally optimal for an arbitrary signal transmission depends on the noise type, and the noise with a high Fisher information is favorable for the signal transmission.

Acknowledgments

This work is sponsored by the NSF of Shandong Province, China (No. ZR2010FM006).

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