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PERIODIC AND APERIODIC STOCHASTIC RESONANCE WITH OUTPUT SIGNAL-TO-NOISE RATIO EXCEEDING THAT AT THE INPUT

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Stochastic resonance (SR) is a nonlinear effect whereby a system is able to improve, via noise addition, the detectability of a signal in noise. SR has been demonstrated with different types of systems and signals where in each case, an appropriate detectability measure is shown improvable at the output of the stochastic resonator when noise is added at its input. A complementary issue, important for practical applications of SR, is the possibility of making the signal detectability at the ouput exceed that at the input when noise is added. We demonstrate this possibility, for both periodic and aperiodic SR, with a simple nonlinear system that we show exactly tractable analytically.

1. Introduction

Stochastic resonance (SR) is a nonlinear effect wherein a system can transmit a signal with improved efficacy when noise is added [Moss *et al.*, 1994; Wiesenfeld & Moss, 1995]. SR has first been reported for bistable dynamic systems transmitting a sinusoidal signal [Benzi *et al.*, 1981; Gammaitoni *et al.*, 1998]. SR has then been extended to other nonlinear systems, including dynamic as well as static nonlinearities, and to other signals, including periodic nonsinusoidal or aperiodic signals [Stocks *et al.*, 1993; Anishchenko *et al.*, 1992, 1994; Collins *et al.*, 1995; Chapeau-Blondeau & Godivier, 1997]. In each case, an appropriate detectability measure [Wiesenfeld & Moss, 1995; Collins *et al.*, 1995; Heneghan *et al.*, 1996; Neiman *et al.*, 1996; Bulsara & Zador, 1996; Godivier & Chapeau-Blondeau, 1998] of the coherent signal in the signal-plus-noise mixture, is shown to be improvable at the output of the stochastic resonator when noise is added at its input. Yet, another important question, especially relevant for applications, is the possibility in SR of making the signal detectability at the ouput exceed that at the input when noise is added.

This issue is considered, for periodic SR when the detectability measure is a signal-to-noise ratio (SNR) at the frequency of the coherent signal [Wiesenfeld & Moss, 1995], with threshold devices [Jung, 1994, 1995], where the output SNR is reported to always remain below the input SNR for all conditions tested. Later, proofs were given that, for periodic SR, the output SNR can never exceed the input SNR [Dykman *et al.*, 1995; DeWeese & Bialek, 1995]. But strictly, this property is proved only for the small-signal limit and with Gaussian noise, what is far from exhausting the conditions under which SR can occur.

Kiss [1996] and Loerincz *et al.* [1996] circumvent the conditions of these proofs of impossibility (especially the small-signal regime), by considering an input train of rectangular pulses transmitted by a unidirectional level-crossing detector [Gingl *et al.*, 1995]. For aperiodic input pulses, with a SNR in the frequency domain differing from the conventional

SNR introduced above, Kiss [1996] uses an approximate treatment to show a SNR larger at the output than at the input. For periodic input pulses, with the conventional SNR, Loerincz *et al.* [1996] use analog and numerical simulations to show a SNR larger at the output than at the input. A comparable result has been reported with a Schmitt trigger in place of the unidirectional level-crossing detector [Khovanov & Anishchenko, 1997].

Despite these positive results, the effective possibility of a detectability measure larger at the output than at the input in SR, does not yet seem fully appreciated, as suggested by the picture emanating from [Dykman & McClintock, 1998]. To contribute to firmly establishing this property, we consider here a simple nonlinear system that we show exactly tractable analytically for both periodic and aperiodic SR. We demonstrate conditions yielding a SNR larger at the output than at the input in periodic SR. In the same way for aperiodic SR, we introduce a natural detectability measure and demonstrate that it can be larger at the output than at the input.

Compared to our present approach, the previous demonstrations have used approximate treatments or simulations and are limited to a Gaussian noise [Kiss, 1996; Loerincz *et al.*, 1996] or a periodic input [Khovanov & Anishchenko, 1997; Chapeau-Blondeau, 1997a]. By contrast, our present demonstration, through the use of a different stochastic resonator, realizes an exact analytical treatment, that can deal with Gaussian or non-Gaussian (arbitrarily distributed) noise, and, for periodic SR, with a periodic input of arbitrary waveform. Especially, periodic SR with a SNR larger at the output than at the input, is shown possible on a periodic pulse train or on a sinewave. Also, for the same stochastic resonator, both periodic and aperiodic SR are implemented and described by our exact analytical approach.

We focus here on the simple stochastic resonator, also considered in [Gammaitoni, 1995], formed by the static nonlinearity

$$g(u) = \begin{cases} -1 & \text{for } u < -\theta, \\ 0 & \text{for } -\theta \le u \le \theta, \\ 1 & \text{for } u > \theta, \end{cases} \qquad (1)$$

with the threshold $\theta > 0$.

2. Periodic Stochastic Resonance

First, we consider the case where the coherent signal is $s(t)$, periodic with the period T_s. It is corrupted by $\eta(t)$ a stationary white noise with the probability density function $f_\eta(u)$ and the cumulative distribution function $F_\eta(u) = \int_{-\infty}^{u} f_\eta(u')du'$. The signal-plus-noise mixture $s(t) + \eta(t)$ is input onto the nonlinearity of Eq. (1) to deliver the output $y(t) = g[s(t) + \eta(t)]$.

In the power spectral density (PSD) of the noisy input $s(t) + \eta(t)$, the coherent part at frequency n/T_s is measured by the power $|S_n|^2$ contained in the coherent spectal line at n/T_s, with the Fourier coefficient

$$S_n = \frac{1}{T_s}\int_0^{T_s} s(t)\exp\left(-in\frac{2\pi}{T_s}t\right)dt\,, \qquad (2)$$

and the incoherent statistical fluctuations in the input $s(t) + \eta(t)$, which control the continuous noise background in the input PSD, are measured by the variance σ_η^2 of the input white noise $\eta(t)$.

In the same way, in the PSD of the output $y(t)$, the coherent part at frequency n/T_s is measured by the power $|\overline{Y}_n|^2$ contained in the coherent spectal line at n/T_s, with the Fourier coefficient

$$\overline{Y}_n = \frac{1}{T_s}\int_0^{T_s} \mathrm{E}[y(t)]\exp\left(-in\frac{2\pi}{T_s}t\right)dt \qquad (3)$$

of the T_s-periodic nonstationary output mean $\mathrm{E}[y(t)]$ [Chapeau-Blondeau & Godivier, 1997; Chapeau-Blondeau, 1997a]. The incoherent statistical fluctuations in the output $y(t)$, which control the continuous noise background in the output PSD, are measured by the average

$$\overline{\mathrm{var}(y)} = \frac{1}{T_s}\int_0^{T_s} \mathrm{var}[y(t)]dt \qquad (4)$$

of the nonstationary output variance $\mathrm{var}[y(t)]$ [Chapeau-Blondeau & Godivier, 1997; Chapeau-Blondeau, 1997a].

The ratio of the output SNR to the input SNR follows, for the coherent component at frequency n/T_s, as

$$\frac{R_{\mathrm{out}}}{R_{\mathrm{in}}} = \frac{|\overline{Y}_n|^2/\overline{\mathrm{var}(y)}}{|S_n|^2/\sigma_\eta^2}\,. \qquad (5)$$

For a static nonlinearity defined by any function $g(u)$, we have

$$\mathrm{E}[y(t)] = \int_{-\infty}^{+\infty} g(u)f_\eta[u - s(t)]du\,, \qquad (6)$$

and

$$\mathrm{var}[y(t)] = \int_{-\infty}^{+\infty} g^2(u)f_\eta[u-s(t)]du - \mathrm{E}^2[y(t)]\,. \qquad (7)$$

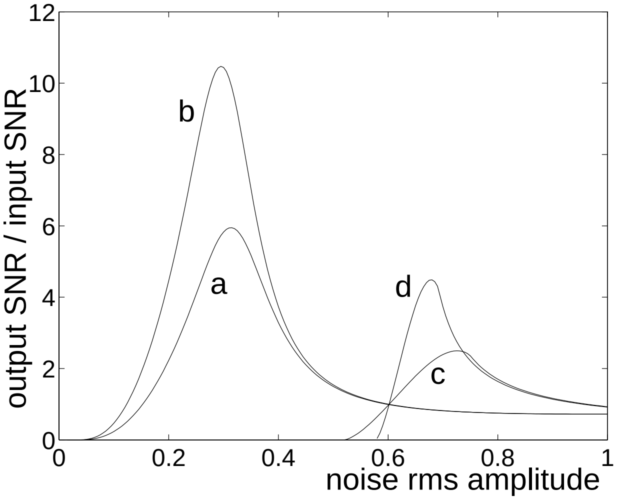

Fig. 1. Ratio of the output SNR to the input SNR $R_{\text{out}}/R_{\text{in}}$ from Eqs. (5)–(9) at the fundamental frequency $1/T_s$ when $\theta = 1.1$, as a function of the rms amplitude of the zero-mean noise $\eta(t)$. In (a) and (b) $\eta(t)$ is Gaussian and the T_s-periodic input is the bipolar pulse train $s(t) = 1$ for $t \in [0,\, T < T_s/2]$, $s(t) = -1$ for $t \in [T_s/2,\, T + T_s/2]$, and $s(t) = 0$ elsewhere over the period T_s. In (c) and (d) $\eta(t)$ is uniform and $s(t) = A_s \sin(2\pi t/T_s)$. Also in (a) $T/T_s = 4 \times 10^{-3}$, in (b) $T/T_s = 2 \times 10^{-3}$, in (c) $A_s = 0.2$, in (d) $A_s = 0.1$.

For the nonlinearity of Eq. (1), these expressions reduce to

$$\mathrm{E}[y(t)] = 1 - F_\eta[\theta - s(t)] - F_\eta[-\theta - s(t)]\,, \quad (8)$$

and

$$\begin{aligned}\mathrm{var}[y(t)] = &\{1 - F_\eta[\theta - s(t)]\}F_\eta[\theta - s(t)] \\ &+ \{1 - F_\eta[-\theta - s(t)]\}F_\eta[-\theta - s(t)] \\ &+ 2\{1 - F_\eta[\theta - s(t)]\}F_\eta[-\theta - s(t)]\,. \quad (9)\end{aligned}$$

Equations (8) and (9) inserted in Eqs. (3) and (4) lead to an exact expression for the ratio $R_{\text{out}}/R_{\text{in}}$ of Eq. (5), for the nonlinearity of Eq. (1) transmitting an arbitrary T_s-periodic input $s(t)$ with the noise influence conveyed by $F_\eta(u)$. Figure 1 illustrates that conditions can be easily found for $s(t)$ and $F_\eta(u)$ that yield $R_{\text{out}}/R_{\text{in}} > 1$.

3. Aperiodic Stochastic Resonance

Now, we consider the case where the coherent signal is formed by the discrete-time sequence s_j, where s_j is a binary random variable assuming the value $+1$ or -1 respectively with the probabilities p_1 and $p_{-1} = 1 - p_1$. The sequence s_j is corrupted by a noise η_j, where η_j is a continuous random variable with the cumulative distribution function $F_\eta(u) = \Pr\{\eta_j \le u\}$. The successive realizations of s_j are independent and identically distributed, and the same for η_j; also s_j and η_j are independent.

The input signal-plus-noise mixture $s_j + \eta_j$ is transmitted by the nonlinearity of Eq. (1) and produces the output $y_j = g(s_j + \eta_j)$. We are dealing with a nonlinear memoryless binary channel with erasure [Cover & Thomas, 1991], where the input $s_j = \pm 1$ can be received by $y_j = \pm 1$ possibly with an error, or erased when $y_j = 0$.

We are now interested in defining a measure of the information contained in the output y_j about the input s_j. This measure will be the analog of the ouput SNR in the transmission of a periodic input signal. A meaningful measure is provided [Cover & Thomas, 1991] by the Shannon input–output mutual information $I(s_j; y_j) = H(y_j) - H(y_j|s_j)$. With $h(u) = -u \log_2(u)$, the output entropy is $H(y_j) = \sum_y h(\Pr\{y_j = y\})$, and the input–output conditional entropy is $H(y_j|s_j) = \sum_s \Pr\{s_j = s\} \sum_y h(\Pr\{y_j = y | s_j = s\})$.

The probabilities involved in the entropies can be explicitly derived. For instance, one has the probability $p_{1,-1} = \Pr\{y_j = 1 \,|\, s_j = -1\}$ which is also $\Pr\{s_j + \eta_j > \theta \,|\, s_j = -1\}$, amounting to $\Pr\{\eta_j > \theta + 1\} = 1 - F_\eta(\theta + 1)$. With similar rules one arrives at:

$$p_{1,-1} = 1 - F_\eta(\theta + 1)\,, \quad (10)$$

$$p_{1,1} = 1 - F_\eta(\theta - 1)\,, \quad (11)$$

$$p_{-1,1} = F_\eta(-\theta - 1)\,, \quad (12)$$

$$p_{-1,-1} = F_\eta(-\theta + 1)\,, \quad (13)$$

$$p_{0,1} = [1 - F_\eta(-\theta - 1)]F_\eta(\theta - 1)\,, \quad (14)$$

$$p_{0,-1} = [1 - F_\eta(-\theta + 1)]F_\eta(\theta + 1)\,. \quad (15)$$

The entropies follow as

$$\begin{aligned}H(y_j \,|\, s_j) = &\; p_1[h(p_{1,1}) + h(p_{0,1}) + h(p_{-1,1})] \\ &+ (1 - p_1)[h(p_{1,-1}) \\ &+ h(p_{0,-1}) + h(p_{-1,-1})]\,, \quad (16)\end{aligned}$$

and

$$\begin{aligned}H(y_j) = &\; h[p_{1,1}p_1 + p_{1,-1}(1 - p_1)] \\ &+ h[p_{-1,1}p_1 + p_{-1,-1}(1 - p_1)] \\ &+ h[p_{0,1}p_1 + p_{0,-1}(1 - p_1)]\,. \quad (17)\end{aligned}$$

Equations (10)–(17) allow a complete calculation of the mutual information $I(s_j; y_j)$. It is possible to go further in the characterization of the nonlinear transmission, at the price of an assumption that is not severely restrictive, i.e. the assumption of an even probability density function for the noise η_j. In this case, we are in the presence of a *symmetric* memoryless discrete channel, for which the mutual information $I(s_j; y_j)$ is invariant under the exchange of p_1 and $p_{-1} = 1 - p_1$. Since for any memoryless discrete channel the input–output mutual information is a concave function of the input probability distribution [Cover & Thomas, 1991], we conclude that for η_j with an even probability density, $I(s_j; y_j)$ reaches its maximum for $p_1 = 1 - p_{-1} = 1/2$. This maximum defines the information capacity C_{out} between the output y_j and the input s_j. Equations (16) and (17), with $p_1 = 1/2$, allow then a complete calculation of the information capacity C_{out}, as a function of the noise properties conveyed by $F_\eta(u)$ via Eqs. (10)–(15).

In the regime where the threshold $\theta > 1$, the input signal $s_j = \pm 1$ is unable by itself to trigger transitions in the output y_j, and C_{out} is strictly zero in the absence of the noise. Equations (16) and (17) then show that addition of noise enables the transmission of information resulting in $C_{\text{out}} > 0$, with C_{out} culminating at a maximum value for an optimal noise level. This nonmonotonic evolution of C_{out} with the noise level is the analog of the evolution of the output SNR in periodic SR. This noise-assisted transmission of an aperiodic signal, measured by a resonant information capacity C_{out}, is another form of SR, also reported in a slightly different channel in [Chapeau-Blondeau, 1997b]. But here we are not interested in observing the signal detectability at the output (measured by C_{out}) being improved by noise addition. We are interested in the possibility of making, through noise addition, the signal detectability at the output of the stochastic resonator larger than at its input.

Information about the coherent input signal s_j can be extracted directly from the input signal-plus-noise mixture $s_j + \eta_j$ by deciding that s_j is 1 when $s_j + \eta_j$ is found > 0 and s_j is -1 when $s_j + \eta_j$ is found < 0. In this case, for η_j with an even probability density, the decision process of s_j from $s_j + \eta_j$ represents a symmetric binary channel, for which it is known [Cover & Thomas, 1991] that the maximum information obtained from $s_j + \eta_j$ about s_j is measured by the information capacity (defined as the maximum mutual information obtained when $p_1 = 1 - p_{-1} = 1/2$):

$$C_{\text{in}} = 1 + p \log_2(p) + (1 - p) \log_2(1 - p) \,. \quad (18)$$

In Eq. (18), p is the probability of correct decision, i.e. the probability of deciding from $s_j + \eta_j$ that the coherent input s_j is 1 when indeed it is 1, equal to the probability of deciding a coherent input -1 when it is -1. This probability p is simply expressable as $p = F_\eta(1)$, which provides an explicit relation, through Eq. (18), of the measure C_{in} to the properties of the noise η_j.

At the input, the detectability of the signal s_j in the input signal-plus-noise mixture $s_j + \eta_j$, as characterized by C_{in}, tends to be perfect when the noise η_j vanishes. This is measured by C_{in} that goes to 1 in the absence of the noise. This behavior is the analog of the input SNR in periodic SR, that goes to infinity at zero noise. Equation (18) also shows that when the level of the noise η_j is gradually raised above zero, C_{in} experiences a monotonic decay from 1 down to zero for large noise levels, much like the input SNR in periodic SR.

But the issue we want to address here is the possibility of obtaining $C_{\text{out}} > C_{\text{in}}$ for a certain range of the noise level, just like we were seeking a SNR larger at the output than at the input in periodic SR. This possibility can indeed be verified, as demonstrated by Fig. 2 which represents illustrative conditions yielding a ratio $C_{\text{out}}/C_{\text{in}} > 1$.

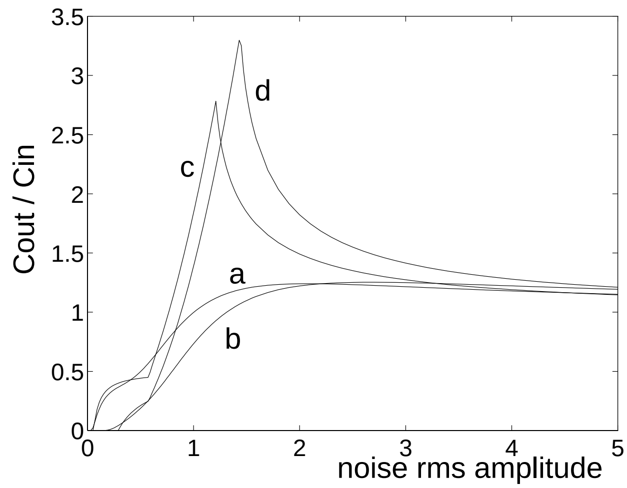

Fig. 2. Ratio of the information capacities $C_{\text{out}}/C_{\text{in}}$ as a function of the rms amplitude of the zero-mean noise η_j. In (a) and (b) η_j is Gaussian; in (c) and (d) η_j is uniform. Also in (a) and (c) $\theta = 1.1$; in (b) and (d) $\theta = 1.5$.

4. Discussion and Conclusion

For periodic SR, a ratio $R_{\rm out}/R_{\rm in} > 1$ was shown possible here, simply by circumventing the conditions of the proof of impossibility [Dykman *et al.*, 1995; DeWeese & Bialek, 1995] that required a small signal and Gaussian noise. Either with a signal that is not small (relative to the noise rms amplitude or to the scale set by the threshold θ), or with a non-Gaussian noise, the simple stochastic resonator of Eq. (1) is able to produce $R_{\rm out}/R_{\rm in} > 1$, as demonstrated in Fig. 1. In other conditions, Chapeau-Blondeau [1997a] arrived at an identical conclusion for periodic SR. The same issue for aperiodic SR is resolved here for the first time, with the results of Fig. 2 showing $C_{\rm out}/C_{\rm in} > 1$. The present results constitute a unique framework, by establishing a simple stochastic resonator completely tractable analytically for both periodic and aperiodic SR, and able to demonstrate that SR can produce a detectability measure of the coherent signal (either periodic or aperiodic) that is better at the ouput than at the input of the stochastic resonator. It is to note that the conditions of the present report are not limitative, merely illustrative. Especially, the presence of the double threshold in the nonlinearity of Eq. (1) is not necessary to obtain a detectability measure better at the output than at the input. The same could be obtained, in appropriate conditions, with a single threshold nonlinearity, similar to a neuron response. In this respect, pulse-like signals with Gaussian noise, as in Figs. 1 and 2, bear similarities with a train of postsynaptic potentials added to Gaussian membrane noise to reach a neuron threshold. Therefore, contrary to the view in [Dykman & McClintock, 1998], the detectability of a noisy signal after transduction by a neural cell can be expected to be better than that of the incoming signal from the environment, with an improvement which is maximized at the optimal noise level prescribed by the SR effect.

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