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Noise-assisted signal transmission in a nonlinear electronic comparator: Experiment and theory

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Abstract

A periodic signal superposed to a white noise is input onto a nonlinear two-state threshold comparator. The output signal contains, embedded in random fluctuations originating in the input noise, a part of correlation with the periodic input. We show that a regime exists where this part of correlation can be enhanced by means of an increase of the input noise level. This is the phenomenon of stochastic resonance, whereby the noise becomes beneficial to the transmission of a coherent signal, and where an increase of the noise can result in improved performances. We experimentally demonstrate this property in an electronic circuit. We then develop a complete theoretical analysis of this effect of noise-assisted signal transmission. This simple electronic implementation of a stochastic resonator, together with its theoretical analysis, constitute a unique framework for further investigations on the nonlinear phenomenon of stochastic resonance and its implications for signal processing. © 1997 Published by Elsevier Science B.V.

Zusammenfassung

Ein periodisches Signal, das weißem Rauschen überlagert ist, wird als Eingangssignal einer nichtlinearen Vergleichsschaltung mit zwei Zuständen betrachtet. Das Ausgangssignal enthält einen Anteil, der mit dem periodischen Eingangssignal korreliert ist, und der in zufällige Schwankungen eingebettet ist, die vom Eingangsrauschen herrühren. Wir zeigen, daß es einen Arbeitsbereich gibt, bei dem dieser Korrelationsanteil durch ein Anheben des Eingangsrauschens verstärkt werden kann. Dies ist das Phänomen der stochastischen Resonanz, wobei das Rauschen für das Übertragen eines kohärenten Signals günstig ist, und wobei ein Anwachsen des Rauschens eine verbesserte Leistungsfähigkeit erzeugt. Diese Eigenschaft demonstrieren wir experimentell für eine elektronische Schaltung. Danach entwickeln wir eine vollständige theoretische Analyse dieses Effektes einer rauschunterstützten Signalübertragung. Diese einfache elektronische Realisierung eines stochastischen Resonators bildet zusammen mit seiner theoretischen Analyse eine allgemeine Grundlage für weitere Untersuchungen der stochastischen Resonanz und ihrer Bedeutung für die Signalverarbeitung. © 1997 Published by Elsevier Science B.V.

Résumé

Un signal périodique superposé à un bruit blanc est appliqué à l'entrée d'un comparateur non linéaire à seuil à deux états. Le signal de sortie contient, noyée dans des fluctuations aléatoires provenant du bruit d'entrée, une part de

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corrélation avec l'entrée périodique. Nous montrons qu'il existe un régime où cette part de corrélation peut être augmentée au moyen d'une augmentation du niveau de bruit en entrée. Il s'agit du phénomène de résonance stochastique, par lequel le bruit devient avantageux pour la transmission d'un signal cohérent, et où une augmentation du bruit peut conduire à des performances accrues. Nous démontrons expérimentalement cette propriété avec un circuit électronique. Nous développons ensuite une analyse théorique complète de cet effet de transmission du signal assistée par le bruit. Cette réalisation électronique simple d'un résonateur stochastique, accompagnée de son analyse théorique, constituent un cadre unique pour des investigations plus avancées sur le phénomène non linéaire de résonance stochastique et ses implications en traitement du signal.

Keywords: Stochastic resonance; Nonlinear system; Threshold comparator; Signal transmission; Noise; Signal-to-noise ratio

1. Introduction

A nonlinear phenomenon, recently introduced under the name of stochastic resonance [4], is now attracting more and more attention from various scientific communities [32]. The description of this nonlinear phenomenon can be cast under various forms which relate to different specific contexts, such as stochastic processes or mechanical motions in potential fields with barriers. In the context of signal processing, for signal transmission by nonlinear systems, this phenomenon can be described as an increase in the signal-to-noise ratio at the output, which is obtained through an increase of the noise level at the input. A periodic signal superposed to a noise is usually input onto the nonlinear system, and the output signal shows a maximum of coherence or correlation with the periodic input when the input noise reaches a sufficient level. Thus, in this nonlinear effect the noise becomes useful to assist the signal transmission.

This paradoxical nonlinear phenomenon is known, since its introduction, under the name of stochastic resonance, although further developments have shown that it does not bear, in every case, all the characteristics of a conventional resonance. Stochastic resonance was first applied in the context of climate dynamics [3, 25]. It has since been experimentally observed in various systems including lasers [31], electron paramagnetic resonance [14], a magnetoelastic pendulum [30], superconducting devices [17], neurons [8] and electronic circuits [10].

The first electronic circuit to reveal stochastic resonance [10] was a Schmitt trigger with both a threshold and hysteretic nonlinearity. More complex electronic circuits have then been proposed [1], sometimes to simulate nonlinear systems known to exhibit stochastic resonance, like Brownian motions governed by the Langevin equation [16], or neurons [5, 24]. Here we introduce an even simpler circuit, under the form of a two-state threshold comparator with no hysteresis, and we experimentally demonstrate that this system is capable of stochastic resonance. To date, this electronic circuit appears to us as the conceptually simplest experimental system that brings together the ingredients for stochastic resonance. In addition, we show that the property of stochastic resonance observed in this circuit lends itself to an exact theoretical description.

The theoretical modeling of stochastic resonance has to cope with a nonlinear and non-stationary context which usually hinders exact descriptions. Theoretical analyses have been proposed, first for dynamic (i.e. with memory) [22, 20, 13, 9] and more recently for static (memoryless) [19, 15, 11] nonlinear systems. These analyses usually rely on approximations, a frequent one being that of a weak and slow periodic input signal. Also in these treatments the hypothesis of a Gaussian noise is often crucial, and the periodic input is restricted to a sinusoid. In constrast here, for our simple electronic circuit that stochastically resonates, we develop an exact and general theoretical treatment which can accommodate an arbitrarily distributed (not necessarily Gaussian) white noise and a periodic input of arbitrary waveform.

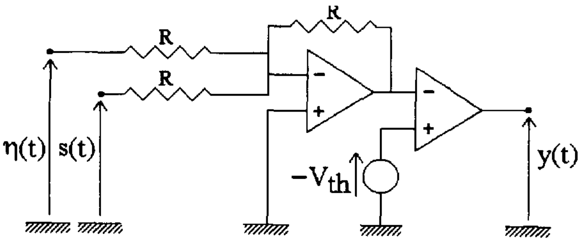

Fig. 1. Experimental electronic circuit implementing Eq. (1).

2. Experimental study

We consider the electronic circuit of Fig. 1 receiving a periodic input signal $s(t)$ and a noise input $\eta(t)$. The first operational amplifier[1] is a linear adder-inverter which delivers $-s(t)-\eta(t)$. Then this signal is applied to the second operational amplifier which operates as a threshold comparator with two output states $+V_{sat}>0$ and $-V_{sat}$. The operation of the circuit is then

$$\text{If } s(t)+\eta(t)>V_{th} \quad \text{then } y(t)=+V_{sat}, \qquad \text{else } y(t)=-V_{sat}. \tag{1}$$

For the experiment we have used a voltage $s(t)$ of period $T_s = 10$ ms, with zero mean and a peak amplitude $V_M = 1.9$ V, $s(t)$ being successively a sinusoid and a square wave half a period at V_M and half a period at $-V_M$. $\eta(t)$ is a zero-mean Gaussian white noise, which in practice was made by an electric noise with a correlation time of less than 0.1 ms, much shorter than the coherent period T_s. The noise generator can be adjusted to change the rms amplitude V_η of $\eta(t)$. We have $V_{sat} = 5$ V.

The threshold V_{th} is set to 2.2 V. In this condition, in the absence of the noise $\eta(t)$, the periodic input $s(t)$ alone is always below the transition threshold V_{th} and the output $y(t)$ remains at $-V_{sat}$. The input $s(t)$ is then invisible from the circuit output $y(t)$. This conforms with a frequent condition associated to stochastic resonance, in which the periodic input alone is usually insufficient to induce a transition of the output. In practice, this can be the situation of a small signal lying below the limit of sensitivity of sensor or transducer devices. We then show that addition of noise on the input renders the periodic input visible in the output $y(t)$, in a more acute manner as the noise is increased, up to an optimal noise level.

To analyze this property, the noise rms amplitude V_η was set at a fixed finite value. The correlation of the output signal $y(t)$ with the periodic input $s(t)$ was then quantified in the following way. $y(t)$ was sampled with a time step $\Delta t = 0.1$ ms and quantized by a 12-bit AD converter. The output autocorrelation function $R_{yy}(\tau) \equiv \langle y(t)y(t+\tau)\rangle$ was estimated by averaging a total of N_{av} products $y(t)y(t+\tau)$, with values of $t \bmod T_s$ uniformly covering the interval $[0, T_s[$ for every value of τ at which $R_{yy}(\tau)$ was estimated. A typical output autocorrelation function $R_{yy}(\tau)$ is depicted in Fig. 2.

A Fourier transform of $R_{yy}(\tau)$ then yielded the output power spectral density $P_{yy}(\nu)$. A typical PSD $P_{yy}(\nu)$ is depicted in Fig. 3. Typically, $P_{yy}(\nu)$ is formed by sharp spectral lines at integer multiples n/T_s of the periodic input frequency $1/T_s$, emerging from a broad-band continuous background. The spectral lines are the mark of the influence of the periodic input $s(t)$ on the output $y(t)$, and the continuous background of the influence of the noise $\eta(t)$ on $y(t)$. In the presence of stochastic resonance, the emergence of the spectral lines above the noise background is most pronounced for an optimal noise level on the input. The effect was measured by a signal-to-noise ratio $\mathrm{SNR}(n/T_s)$ defined as the ratio of the power contained in the spectral line alone at the harmonic n/T_s, to the power contained in the continuous noise background in a given fixed frequency band around $\nu = n/T_s$. The power contained in the spectral line at frequency n/T_s was calculated as the product of the magnitude of the line above the noise background, times the frequency step $\Delta\nu$ of the discrete Fourier transform. The power of the noise background was evaluated in a frequency band of extension $1/T_s$ around $\nu = n/T_s$, as the product of the constant amplitude of the continuous background, times $1/T_s$.

Such an SNR was computed from the experimentally estimated output PSD $P_{yy}(\nu)$, for different harmonics n/T_s. We then studied the evolution of the SNR with the input noise rms amplitude V_η. This evolution is presented in Fig. 4. The results of

[1] We used a dual TL082CP operational amplifier circuit.

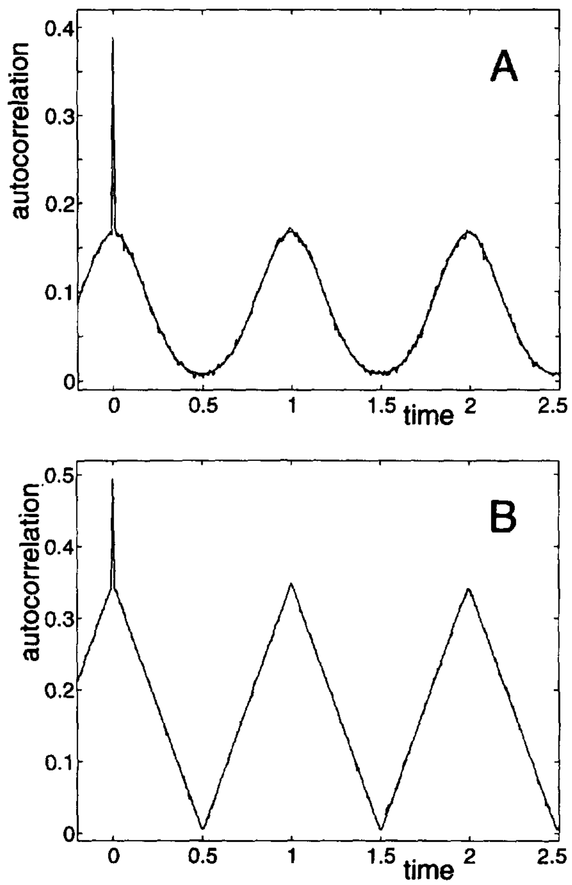

Fig. 2. Output autocorrelation function $R_{yy}(\tau)$ in units V_{sat}^2 as a function of the time lag τ in units T_s. Panel A is with $s(t) = V_M \sin(2\pi t/T_s)$ and panel B with $s(t)$ a square wave half a period T_s at V_M and half a period at $-V_M$. The threshold is $V_{th} = 2.2$ V and $V_M = 1.9$ V. The noise rms amplitude is here $V_\eta = 1.4$ V. In each panel the smooth line is the theoretical expression of Eq. (9), the noisy line is the experimental estimation by averaging $N_{av} = 30\,000$ products $y(t)y(t+\tau)$ for every value of τ.

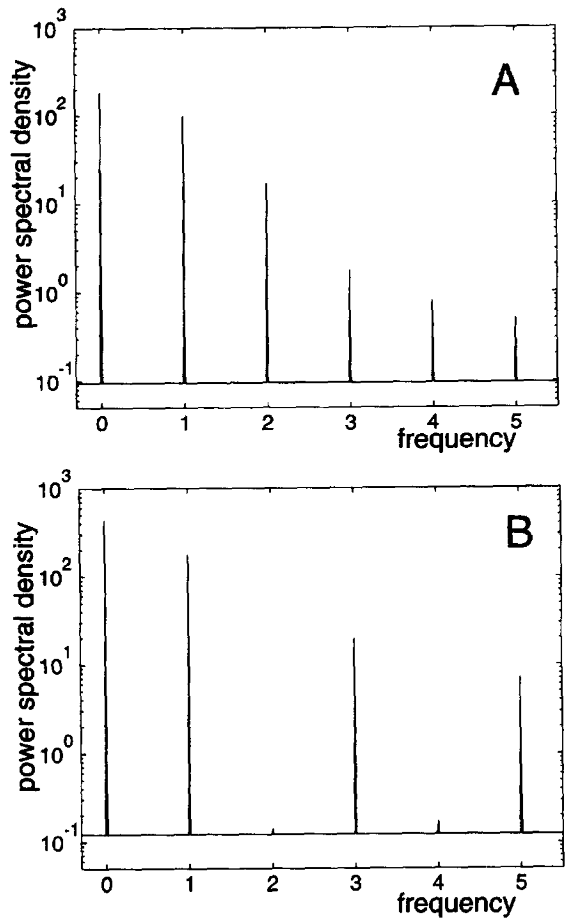

Fig. 3. Output power spectral density $P_{yy}(\nu)$ as a function of the frequency ν in units $1/T_s$, obtained by Fourier transforming the autocorrelation function $R_{yy}(\tau)$ (with $M = 100$). Panel A is with $s(t) = V_M \sin(2\pi t/T_s)$ and panel B with $s(t)$ a square wave half a period T_s at V_M and half a period at $-V_M$. In each panel the smooth line is the theoretical expression of the PSD similar to Eq. (13), the noisy line (quasi indistinguishable) is the experimental PSD deduced by averaging with $N_{av} = 3 \times 10^7$ in the estimation of the autocorrelation R_{yy}.

Fig. 4 show a nonmonotonic variation where the SNR increases with the noise rms amplitude, up to an optimal noise level where the SNR is maximized. This nonmonotonic influence of the noise on the SNR is the signature of stochastic resonance.

3. Theoretical analysis

We consider the system described by Eq. (1) with a periodic $s(t)$ of period T_s and $\eta(t)$ a stationary white noise with the distribution function $F_\eta(u) = \Pr\{\eta(t) \leqslant u\}$. We shall adapt in this section, the theoretical method of description that has been introduced in [6] for a slightly different nonlinear system. First, we seek to compute a statistical autocorrelation function for the output signal $y(t)$. Since y only assumes the values V_{sat} or $-V_{sat}$, the expectation $E[y(t)y(t+\tau)]$, for fixed $\tau \neq 0$ and fixed t, can be expressed from the

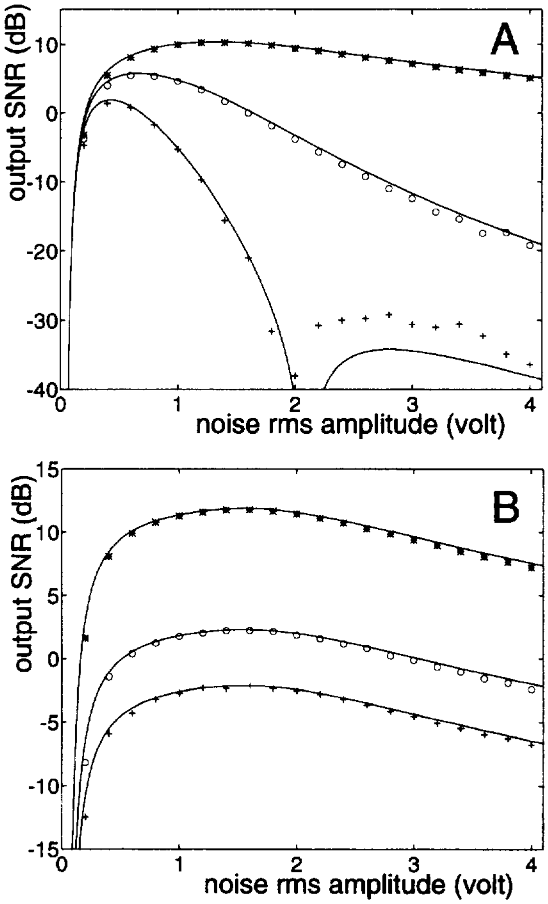

Fig. 4. Output signal-to-noise ratio SNR as a function of the input noise rms amplitude V_η in volts. Panel A is with $s(t) = V_M \sin(2\pi t/T_s)$ and panel B with $s(t)$ a square wave half a period T_s at V_M and half a period at $-V_M$. In each panel the solid line is the theoretical expression of Eq. (14), the sets of discrete data points were obtained experimentally from estimations of the autocorrelation function $R_{yy}(\tau)$ and averaging with $N_{av} = 3 \times 10^7$. In each panel, the SNR is shown at the fundamental frequency $1/T_s$ and at the two subsequent harmonics n/T_s that give the strongest SNR; panel A: (∗) at $1/T_s$, (○) at $2/T_s$ and (+) at $3/T_s$; panel B: (∗) at $1/T_s$, (○) at $3/T_s$ and (+) at $5/T_s$.

probabilities

$$E[y(t)y(t+\tau)] = V_{sat}^2[\Pr\{y(t) = V_{sat}, y(t+\tau) = V_{sat}\} + \Pr\{y(t) = -V_{sat}, y(t+\tau) = -V_{sat}\} - \Pr\{y(t) = V_{sat}, y(t+\tau) = -V_{sat}\} - \Pr\{y(t) = -V_{sat}, y(t+\tau) = V_{sat}\}]. \quad (2)$$

We introduce

$$x(t) = \Pr\{\eta(t) > V_{th} - s(t)\} = 1 - F_\eta[V_{th} - s(t)]. \quad (3)$$

Since $s(t)$ is deterministic and $\eta(t)$ is a *white* noise, $y(t)$ and $y(t+\tau)$ for fixed $\tau \neq 0$ and fixed t, are statistically independent, and Eq. (2) becomes

$$E[y(t)y(t+\tau)] = V_{sat}^2[x(t)x(t+\tau) + [1 - x(t)][1 - x(t+\tau)] - x(t)[1 - x(t+\tau)] - [1 - x(t)]x(t+\tau)], \quad (4)$$

which simplifies into

$$E[y(t)y(t+\tau)] = V_{sat}^2[4x(t)x(t+\tau) + 1 - 2x(t) - 2x(t+\tau)]. \quad (5)$$

And for $\tau = 0$, one has

$$E[y(t)y(t+\tau)] = V_{sat}^2. \quad (6)$$

Both $x(t)$ and $x(t+\tau)$ are periodic in t and τ with period T_s. Because of the periodic coherent modulation introduced by $s(t)$, the random signal $y(t)$ is nonstationary, yet it is cyclostationary with period T_s [26]. It is possible to construct a 'stationary' autocorrelation function $R_{yy}(\tau)$ for $y(t)$ through a proper time averaging of $E[y(t)y(t+\tau)]$ over an interval T_s, when t, or t mod T_s, uniformly covers $[0, T_s[$.

To avoid difficulties attached to the idealized notion of a white noise, and to have the possibility of direct numerical evaluation of every relevant quantity of the theoretical model, especially for purpose of comparison with the experiment, we choose now to move to the context of discrete-time signals. The time scale is thus discretized with a step $\Delta t \ll T_s$, and such that $T_s = N\Delta t$. Now in practice, the white noise $\eta(t)$ only need be a noise with a correlation time shorter than Δt.

We define the stationary autocorrelation function as

$$R_{yy}(k\Delta t) = \frac{1}{N}\sum_{j=0}^{N-1} E[y(j\Delta t)y(j\Delta t + k\Delta t)]. \quad (7)$$

Eqs. (5) and (6) can be combined into a single equation for every $t = j\,\Delta t$ and $\tau = k\,\Delta t$, which reads

$$E[y(j\,\Delta t)y(j\,\Delta t + k\,\Delta t)] = 4V_{\text{sat}}^2[x(j\,\Delta t) - x^2(j\,\Delta t)]\hat{\delta}(k\,\Delta t) + V_{\text{sat}}^2[4x(j\,\Delta t)x(j\,\Delta t + k\,\Delta t) + 1 - 2x(j\,\Delta t) - 2x(j\,\Delta t + k\,\Delta t)], \quad (8)$$

with $\hat{\delta}(k\,\Delta t) = 1$ for $k = 0$, and $\hat{\delta}(k\,\Delta t) = 0$ otherwise. For establishing the expression of Eq. (8), we simply applied Eq. (5) everywhere (as it appears in the second term of the right-hand side of Eq. (8)), except in $\tau = 0$. In $\tau = k\,\Delta t = 0$, Eq. (5) evaluates to $E[y(j\,\Delta t)y(j\,\Delta t)] = V_{\text{sat}}^2 + 4V_{\text{sat}}^2[x^2(j\,\Delta t) - x(j\,\Delta t)]$ which is in excess by $4V_{\text{sat}}^2[x^2(j\,\Delta t) - x(j\,\Delta t)]$ from the correct value $E[y(j\,\Delta t)y(j\,\Delta t)] = V_{\text{sat}}^2$ of Eq. (6). This value in excess has just been substracted in Eq. (8), by means of the first term on its right-hand side, which operates only when $k\,\Delta t = 0$.

Performing on Eq. (8) the time average defined by Eq. (7) then leads to

$$R_{yy}(k\,\Delta t) = 4V_{\text{sat}}^2(\bar{x} - \overline{x^2})\hat{\delta}(k\,\Delta t) + \frac{1}{N}\sum_{j=0}^{N-1} 4V_{\text{sat}}^2\, x(j\,\Delta t)x(j\,\Delta t + k\,\Delta t) + V_{\text{sat}}^2(1 - 4\bar{x}), \quad (9)$$

with the time average

$$\bar{x} = \frac{1}{N}\sum_{j=0}^{N-1} x(j\,\Delta t), \quad (10)$$

and a similar definition for the average $\overline{x^2}$. We note that since $F_\eta(u)$ varies between 0 and 1, for any non-identically zero or one $x(t)$, one always has $\bar{x} - \overline{x^2} > 0$.

In order to proceed into the frequency domain, the Fourier coefficients of the deterministic periodic signal $x(j\,\Delta t)$ are introduced as

$$X_n = \frac{1}{N}\sum_{j=0}^{N-1} x(j\,\Delta t)\exp\left(-\mathrm{i}2\pi\frac{jn}{N}\right). \quad (11)$$

We define the discrete Fourier transform of R_{yy}, over a time interval of an integer number $2M$ of periods T_s, as

$$\mathrm{DFT}[R_{yy}(k\,\Delta t)] = \sum_{k=-MN}^{MN-1} R_{yy}(k\,\Delta t)\exp\left(-\mathrm{i}2\pi\frac{kl}{2MN}\right), \quad (12)$$

which affords a frequency resolution $\Delta\nu = 1/(2MN\,\Delta t)$.

The autocorrelation function of Eq. (9) is formed by the superposition of three components: (1) a pulse at the origin with magnitude $4V_{\text{sat}}^2(\bar{x} - \overline{x^2})$, (2) a periodic component with period T_s (the term of the Σ sign in Eq. (9)), (3) a constant $V_{\text{sat}}^2(1 - 4\bar{x})$. The Fourier transform of R_{yy} defines the output power spectral density P_{yy}, which will then be formed by the Fourier-transformed three components, giving (1) a constant background with magnitude $4V_{\text{sat}}^2(\bar{x} - \overline{x^2})$, (2) a series of spectral lines at integer multiples of $1/T_s$, (3) an additional line at zero-frequency with weight $V_{\text{sat}}^2(1 - 4\bar{x})$. The explicit calculation of this Fourier transform is developed in Appendix A. For the output PSD at integer multiples n/T_s of the coherent frequency, with $n \neq 0$, one arrives at

$$P_{yy}\left(\frac{n}{T_s}\right) = 4V_{\text{sat}}^2(\bar{x} - \overline{x^2}) + 2MN \times 4V_{\text{sat}}^2 X_n X_n^*. \quad (13)$$

When the horizon $M \to +\infty$, the magnitude of the spectral lines above the continuous noise background, tends to infinity. This type of form of the PSD is typical for the output of a stochastically resonant system. The power contained in the spectral line at n/T_s of magnitude $2MN \times 4V_{\text{sat}}^2 X_n X_n^*$ is then $2MN \times 4V_{\text{sat}}^2 X_n X_n^* \times \Delta\nu = 4V_{\text{sat}}^2|X_n|^2/\Delta t$. The power in the continuous noise background of constant magnitude $4V_{\text{sat}}^2(\bar{x} - \overline{x^2})$ contained in the frequency band $1/T_s = 1/(N\Delta t)$, is $4V_{\text{sat}}^2(\bar{x} - \overline{x^2})/(N\Delta t)$. The ratio of these two powers defines the output signal-to-noise ratio, at frequency n/T_s, which follows as

$$\mathrm{SNR}\left(\frac{n}{T_s}\right) = \frac{N|X_n|^2}{\bar{x} - \overline{x^2}}. \quad (14)$$

The present theoretical analysis provides expressions for the output autocorrelation function, the output PSD and the SNR at different harmonics, for an arbitrary noise distribution and an arbitrary waveform of the periodic input $s(t)$. This is a unique feature for a system capable of stochastic resonance, which is not offered with other electronic implementations [10, 1, 16, 5, 24].

4. Comparison between theory and experiment

Figs. 2–4 present the comparison between the theory and the experiments for the case of a Gaussian noise $\eta(t)$ and a sinusoidal or square-wave periodic input $s(t)$. The numerical expressions derived in the theory of Section 3 are exact, provided the correlation time of the noise $\eta(t)$ is shorter than Δt, and this was realized in the experiments. As a consequence, a very good agreement is obtained between the theory and the experimental results. The main deviation between the two is introduced by the accuracy at which the autocorrelation function $R_{yy}(\tau)$ can be experimentally estimated, by averaging terms $y(t)y(t + \tau)$. If a sufficient number of such terms are averaged, the accuracy of the estimation can be made very (arbitrarily) small. For the experimental $R_{yy}(\tau)$ depicted in Fig. 2, only $N_{av} = 30\,000$ terms $y(t)y(t + \tau)$ were averaged for each value of $\tau = k\,\Delta t$ where $R_{yy}(\tau)$ was estimated. The objective here was to purposely keep a sufficient variance to the experimental estimation of $R_{yy}(\tau)$ in order to distinguish it from the theoretical $R_{yy}(\tau)$ in Fig. 2. We have verified that when much more than $N_{av} = 30\,000$ terms are averaged, the experimental and theoretical $R_{yy}(\tau)$ become almost indistinguishable. For the rest of the comparison between theory and experiment N_{av} was set to 3×10^7.

From here, the continuation of the comparison between theory and experiment involves only 'mechanical' Fourier transform calculations which are performed numerically in the same way. As a result, the same very good agreement is preserved between the theoretical and experimental PSDs depicted in Fig. 3, since the corresponding autocorrelation functions almost perfectly agree.

Finally, Fig. 4 compares the SNRs which are calculated from the PSDs. The SNRs are shown at the fundamental frequency $1/T_s$ and at the two subsequent harmonics n/T_s that give the strongest SNRs. A very good agreement persists between theory and experiment, which slowly degrades for very low values of the SNR where the signal is very weak, close to the limit of accuracy of the measurements. Both the experimental and theoretical SNRs display the nonmonotonic variation with the input noise level that is characteristic of stochastic resonance, and that expresses that an increase in the input noise level can favor the transmission of the periodic coherent input.

5. Discussion

The phenomenon of noise-assisted signal transmission through stochastic resonance has been experimentally demonstrated in a very simple two-state nonlinear electronic circuit. The same type of phenomenon was reported before in electronic circuits with more complex nonlinearities [10, 16, 5, 1, 24]. To date, the present system stands for the simplest electronic circuit that has been shown capable of stochastic resonance.

In addition, we have developed an exact theoretical analysis which accounts for the property of stochastic resonance. Especially, the theorey can deal with an input noise with arbitrary distribution (not restricted to the Gaussian), and with a coherent periodic input of arbitrary wave form. These features allow the study of the influence of the noise distribution and periodic wave form to optimize the stochastic resonance effect, although this question has not been explicitly addressed here. To date, very few theoretical models exist in which the property of stochastic resonance lends itself to an exact description, and among the approximate models that exist many are restricted to a Gaussian noise and a sinusoidal periodic input. It is a remarkable, and rare, property of the present model to offer this exact and general description.

Among the various nonlinear systems (dynamic or static) that were reported to exhibit stochastic resonance, the threshold static nonlinearities considered in [11, 12] bear the closest similarities with our simple system of Eq. (1). Yet these systems in [11, 12] can be viewed as conceptually slightly more complex since they involve more than two output states or more than one threshold. The study in [11, 12] also introduces arbitrary noise distributions, but it subsequently offers only an approximate theoretical treatment, especially with only an approximate expression for the SNR. Also in [11, 12] the treatment is not developed for an arbitrary wave form of the periodic input, but the focus is placed on a sinusoidal input. And finally, no experimental implementation is studied in [11, 12].

The stochastic resonance effect can be contrasted with another nonlinear effect, known as dithering, where the noise plays a useful role in signal transmission. Dithering [29, 28, 11] is a technique used in analog-to-digital conversion, whose purpose is to limit the distortions to the statistical properties of a signal that are induced by its quantization. The technique consists in the addition of a noise to the analog signal prior to quantization. The optimal noise must be uniform over an interval equal to the quantization step, and it allows the correct recovery, on the AD converter output, of the first statistical moment of the input signal. Noise-induced threshold crossings are involved in dithering as well as in the presently reported stochastic resonance effect. However, the purpose is different between dithering and stochastic resonance. In dithering, the aim is to recover, in the random output signal, the stationary mean of an input signal. In stochastic resonance, the aim is to recover, in the random output, a time-periodic component showing a maximum of correlation with a periodic input.

As we saw, the theoretical treatment of Section 3 in the framework of discrete-time signals, is exact within the hypothesis of a white noise $\eta(t)$. In practice, a physically realizable 'white' noise, as the one we used in the experiment, has a small, but not strictly zero, correlation time τ_c. When $\tau_c < \Delta t$, the physical (analog) output autocorrelation function has a peak of width $\sim\tau_c$ around the origin, whose magnitude in $\tau = 0$ is correctly represented by Eq. (9), but whose exact shape is not described by Eq. (9). The discrete-time treatment we developed in Section 3 with a time step $\Delta t > \tau_c$, allows us to dispense with explicit assumptions concerning the exact shape of this narrow peak of the autocorrelation. The exact shape of this peak of duration $\sim\tau_c$, will start to manifest its influence on the output power spectral density in the frequency range of order $1/\tau_c$. Such high-frequency perturbations will generally leave unaffected the stochastic resonance effect which takes place in the much lower-frequency range $1/T_s$, and which will consequently be accurately described by the present theoretical treatement. This outcome is very well verified in our present comparison of the theory and experiment.

Another possible departure between theory and experiment could arise from the fact that an actual operational amplifier cannot instantly switch its output state, as expressed by Eq. (1). However, the stochastic resonance effect is essentially sensitive to correlations occurring over the time scale T_s set by the periodic coherent input, and which show up in the frequency range $1/T_s$ on the PSD. We have chosen $1/T_s$ in the range of 100 Hz, much below the cut-off frequency of the operational amplifier, what renders the finite switching time at the output of negligible influence on the stochastic resonance effect.

The accuracy of the comparison between theory and experiment is then essentially limited by the accuracy with which the experimental estimation of the output autocorrelation function $R_{yy}(\tau)$ can be performed. If a very long time segment of $y(t)$ is available, this estimation can be done very accurately as it was shown here, and it revealed a very good agreement between theory and experiment.

The present comparison between theoretical and experimental results uses an ergodic hypothesis which can be located in the assumption that the time average of terms $y(t)y(t + \tau)$ experimentally performed over one realization of the signal as explained in Section 2, is equivalent to the theoretical quantity of Eq. (7) involving ensemble averages. The arguments in [23] (p. 60 and following) show that this property follows as a consequence if the ergodicity of the noise $\eta(t)$ is assumed. Thus, at the root, our ergodic hypothesis amounts to assuming the ergodicity of the noise $\eta(t)$. This noise $\eta(t)$ is a standard electric noise for which there is no a priori reason to doubt that the ergodic hypothesis is a valid representation, although this cannot be rigorously proved, a priori. The hypothesis receives a posteriori justification, as the experimental time averages that are realized are found to perfectly match their theoretical predictions derived via ensemble averages.

We have demonstrated here the possibility of noise-assisted transmission of a periodic signal through stochastic resonance. We emphasize, as also suggested by the results of [2], that other types of 'informative' signals can benefit from this noise-assisted transmission, through a modulation of the periodic input. The periodic input can be considered as a high-frequency carrier wave, whose characteristics can be slowly modulated by a low-frequency 'informative' signal. The transmission of the slowly modulated carrier wave will benefit from the stochastic resonance effect, wherein an optimal input noise level maximizes the magnitude of the carrier as it stands out of the noise in the output signal. Then, band-pass filtering of the output signal in the frequency band of the carrier, followed by a demodulation, will complete the transmission achieved with the assistance of the noise through stochastic resonance.

6. Conclusion

We have considered a simple nonlinear system whose response is conditioned by a threshold. A coherent periodic input varying below the threshold is invisible from the output. We have then shown that addition of noise on the input allows the periodic input to become visible at the output. Furthermore, the signal-to-noise ratio at the output increases as the level of the input noise is increased, up to an optimal noise level where the SNR is maximized. This is the property of stochastic resonance, whereby the noise becomes useful to assist the transmission of a coherent signal, and where a sufficient amount of noise is necessary for optimal transmission.

Stochastic resonance is a relatively newly introduced nonlinear phenomenon which is now experiencing an accelerating growth of interest. At the present time, the relevance and interest of stochastic resonance can be spotted along three main lines.

1. Conceptually, stochastic resonance bears an important significance regarding the status of noise, as it shows that situations can be conceived where the noise ceases to be a nuisance to become beneficial to some coherent signal. Most studies so far, have concentrated on this conceptual aspect, by developing theoretical analyses of the many facets of this nonlinear phenomenon.
2. In the natural world, stochastic resonance has been shown to operate in a very important class of natural systems achieving very efficient signal-processing operations: the neural systems [8, 21, 27, 7]. Stochastic resonance can thus be studied as a relevant property for nonlinear signal processing by neurons (neurons are bound to see signals through a threshold).
3. A third line of interest is formed by the technological applications. Stochastic resonance may have applications for sensors or transducers or communication systems operating at their limit of sensitivity, or for the detection of weak signals in noise [18]. Today, the focus is just arriving to this question of testing stochastic resonance for possible technological applications, and this direction largely remains to be explored. The potentialities revealed by stochastic resonance make such explorations quite worthwhile.

The experimental implementation of a simple stochastic resonator with an electronic circuit, together with the theoretical description that is presented here, constitute a unique and potentially useful framework for further investigations on stochastic resonance.

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Appendix A

In this appendix, we detail the steps of the computation of the Fourier transform, defined by Eq. (12), of the autocorrelation function of Eq. (9), which leads to the DSP of Eq. (13). We shall consider, one by one, the transform of the three terms of the right-hand side of Eq. (9). We note that the discrete Fourier transform defined by Eq. (12) operates on a sequence of $2MN$ time samples, and it returns a sequence of $2MN$ frequency samples labeled by the integer $l \in [-MN, MN-1]$ and which is periodic in l with period $2MN$.

The first term of Eq. (9), $4V_{sat}^2(\bar{x} - \overline{x^2})\hat{\delta}(k\,\Delta t)$ is zero for every k, except $k = 0$. The sum over k in (12) then reduces to the term for $k = 0$, to yield the constant $4V_{sat}^2(\bar{x} - \overline{x^2})$ as it appears in Eq. (13).

The third term of Eq. (9), is the constant $V_{sat}^2(1 - 4\bar{x})$, which factorizes in Eq. (12) so as to multiply the sum

$$W = \sum_{k=-MN}^{MN-1} \exp\left(-\mathrm{i}2\pi \frac{kl}{2MN} \right).$$

This sum W is that of a geometric progression with common ratio $\exp[-\mathrm{i}2\pi l/(2MN)]$. It verifies

$$W - W\exp[-\mathrm{i}2\pi l/(2MN)]$$
$$= \exp\left(+\mathrm{i}2\pi \frac{MNl}{2MN} \right) - \exp\left(-\mathrm{i}2\pi \frac{MNl}{2MN} \right) = 0,$$

for any integer l. For $l \in [-MN, MN-1]$, this implies that $W = 0$ for any $l \neq 0$ (since then $\exp[-\mathrm{i}2\pi l/(2MN)] \neq 1$), and when $l = 0$ the sum W sums up to $2MN$. This represents a pulse at the origin in the frequency domain, with magnitude $V_{sat}^2(1 - 4\bar{x}) \times 2MN$.

The second term of Eq. (9), is (we drop the factor Δt in the argument of x and set $V_{sat}^2 = 1$ for ease of notation) $N^{-1}\Sigma_{j=0}^{N-1} 4x(j)x(j+k)$ which is a periodic function in k with period N. We can equivalently rewrite its Fourier transform as

$$\sum_{k=0}^{2MN-1} \frac{1}{N} \sum_{j=0}^{N-1} 4x(j)x(j+k)\exp\left(-\mathrm{i}2\pi \frac{kl}{2MN} \right) = \alpha_2,$$

and seek to evaluate this sum α_2 for $l \in [0, 2MN-1]$. The sums over j and k can be exchanged to yield

$$\alpha_2 = \frac{4}{N} \sum_{j=0}^{N-1} x(j) \sum_{k=0}^{2MN-1} x(j+k)\exp\left(-\mathrm{i}2\pi \frac{kl}{2MN} \right),$$

which is also

$$\alpha_2 = \frac{4}{N} \sum_{j=0}^{N-1} x(j)\exp\left(\mathrm{i}2\pi \frac{jl}{2MN} \right) \times \beta_2,$$

with the sum

$$\beta_2 = \sum_{k=0}^{2MN-1} x(j+k)\exp\left(-\mathrm{i}2\pi \frac{j+k}{2MN} l \right).$$

To evaluate β_2, we split the index k as $k = k'' + Nk'$. The integer $k' \in [0, 2M-1]$ indexes a bin of extension N, and the integer $k'' \in [0, N-1]$ indexes the position beyond an integer number of these bins. The sum β_2 is then

$$\beta_2 = \sum_{k''=0}^{N-1} \sum_{k'=0}^{2M-1} x(j + k'' + Nk') \times \exp\left(-\mathrm{i}2\pi \frac{j + k'' + Nk'}{2MN} l \right).$$

As $x(j)$ is periodic in j with period N, one has $x(j + k'' + Nk') = x(j + k'')$ for any k', allowing the factorization

$$\beta_2 = \sum_{k''=0}^{N-1} x(j+k'')\exp\left(-\mathrm{i}2\pi \frac{j+k''}{2MN} l \right) \times W',$$

with the sum

$$W' = \sum_{k'=0}^{2M-1} \exp\left(-\mathrm{i}2\pi \frac{k'l}{2M} \right).$$

Again W' is the sum of a geometric progression, with common ratio $\exp[-\mathrm{i}2\pi l/(2M)]$, and it verifies

$$W' - W'\exp[-\mathrm{i}2\pi l/(2M)]$$
$$= 1 - \exp(-\mathrm{i}2\pi l) = 0.$$

This implies that $W' = 0$ for any l, except for $l = n \times 2M$, n integer, where $\exp[-\mathrm{i}2\pi l/(2M)] = 1$ and W' sums up to $2M$. This occurs at frequencies $\nu = l\Delta\nu = n/T_s$ which constitute the spectral lines at n/T_s of the T_s-periodic function $x(j\,\Delta t)\;x(j\,\Delta t + k\,\Delta t)$, and whose Fourier transform vanishes anywhere else.

Now for $l = n \times 2M$, and by virtue of Eq. (11), it comes $\beta_2 = NX_n \times 2M$, and finally $\alpha_2 = 4X_n^* X_n \times 2MN$, which completes the derivation of Eq. (13).

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